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Optimised polarimeter configurations for measuring the Stokes parameters of the cosmic microwave background radiation

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Abstract. We present configurations of polarimeters which measure the three Stokes parameters I, Q and U of the Cosmic Microwave Background Radiation with a nearly diagonal error matrix, independent of the global orientation of the polarimeters in the focal plane. These configurations also provide the smallest possible error box volume.

Key words: cosmic microwave background — cosmology: observations — instrumentation: polarimeters — methods: observational — polarisation

1. Introduction

This paper originates from preparatory studies for the Planck satellite mission. This Cosmic Microwave Background (CMB) mapping satellite is designed to be able to measure the polarisation of the CMB in several frequency channels with the sensitivity needed to extract the expected cosmological signal. Several authors (see for instance Rees 1968; Bond & Efstathiou 1987; Melchiorri & Vittorio 1996; Hu & White 1997; Seljak & Zaldarriaga 1998), have pointed out that measurements of the polarisation of the CMB will help to discriminate between cosmological models and to separate the foregrounds. In the theoretical analyses of the polarised power spectra, it is in general assumed (explicitly or implicitly) that the errors are uncorrelated between the three Stokes parameters I, Q and U[1] in the reference frame used to build the polarised multipoles (Zaldarriaga & Seljak 1997; Ng & Liu 1997). However, the errors in the three Stokes parameters will in general be correlated, even if the noise of the three or more measuring polarimeters are not, unless the layout of the polarimeters is adequately chosen. In this paper we construct configurations of the relative orientations of the polarimeters, hereafter called "Optimised Configurations" (OC), such that, if the noise in all polarised bolometers have the same variance and are not correlated, the measurement errors in the Stokes parameters I, Q and U are independent of the direction of the focal plane and decorrelated. Moreover, the volume of the error box is minimised. The properties of decorrelation and minimum error are maintained when one combines redundant measurements of the same point of the sky, even when the orientation of the focal plane is changed between successive measurements. Finally, when combining unpolarised and data from OC's, the resulting errors retain their optimised properties.

In general, the various polarimeters will not have the same levels of noise and will be slightly cross-correlated. Assuming that these imbalances and cross-correlations are small, we show that for OC's the resulting correlations between the errors on I, Q and U are also small and easily calculated to first order. This remains true when one combines several measurements of the same point of the sky, the correlations get averaged but do not cumulate.

Finally, we calculate the error matrix between E and B multipolar amplitudes and show that it is also simpler in OC's.

1.1. General considerations

In the reference frame where the Stokes parameters I, Q, and U are defined, the intensity detected by a polarimeter rotated by an angle α with respect to the x axis is:

$$I_\alpha = \frac{1}{2}(I + Q\,\cos 2\alpha + U\,\sin 2\alpha). \quad (1)$$

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[1] For a definition of the Stokes parameters, see for instance Born & Wolf (1980) page 554, where I, Q, U are noted s_0, s_1, s_2 respectively.

Because polarimeters only measure intensities, angle α can be kept between 0 and π. To be able to separate the 3 Stokes parameters, at least 3 polarised detectors are needed (or 1 unpolarised and 2 polarised), with angular separations different from multiples of $\pi/2$. If one uses $n \geq 3$ polarimeters with orientations $\alpha_p,\ 1 \leq p \leq n$ for a given line of sight, the Stokes parameters will be estimated by minimising the χ^2:

$$\chi^2 = (\mathbf{M} - \mathbf{AS})^T\ \mathbf{N^{-1}}\ (\mathbf{M} - \mathbf{AS}) \tag{2}$$

where $\mathbf{M} = \begin{pmatrix} m_1 \\ \vdots \\ m_p \\ \vdots \\ m_n \end{pmatrix}$ is the vector of measurements, and $\mathbf{N}$ is their $n \times n$ noise autocorrelation matrix. The $n \times 3$ matrix

$$\mathbf{A} = \frac{1}{2} \begin{pmatrix} 1 & \cos 2\alpha_1 & \sin 2\alpha_1 \\ \vdots & \vdots & \vdots \\ 1 & \cos 2\alpha_p & \sin 2\alpha_p \\ \vdots & \vdots & \vdots \\ 1 & \cos 2\alpha_n & \sin 2\alpha_n \end{pmatrix} \tag{3}$$

relates the results of the n measurements to the vector of the Stokes parameters $\mathbf{S} = \begin{pmatrix} I \\ Q \\ U \end{pmatrix}$ in a given reference frame, for instance a reference frame fixed with respect to the focal instrument. If one looks in the same direction of the sky, but with the instrument rotated by an angle ψ in the focal plane, the matrix A is simply transformed with a rotation matrix of angle $2\,\psi$:

$$\mathbf{A} \rightarrow \mathbf{A}\ \mathbf{R}(\psi),\ \text{with } \mathbf{R}(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\psi & \sin 2\psi \\ 0 & -\sin 2\psi & \cos 2\psi \end{pmatrix}. \tag{4}$$

As is well known, the resulting estimation for the Stokes parameters and their variance matrix $\mathbf{V}$ are:

$$\mathbf{S} = \left(\mathbf{A}^T\,\mathbf{N}^{-1}\,\mathbf{A}\right)^{-1}\ \mathbf{A}^T\mathbf{N}^{-1}\mathbf{M},$$

and

$$\mathbf{V} = \left(\mathbf{A}^T\,\mathbf{N}^{-1}\,\mathbf{A}\right)^{-1}. \tag{5}$$

2. Optimised configurations

2.1. The ideal case

If we assume that the measurements m_p $(1 \leq p \leq n)$ have identical and decorrelated errors $(\mathbf{N}_{pq} =< \delta m_p\ \delta m_q >= \sigma_0^2\,\delta_{pq})$, the χ^2 is simply:

$$\chi^2 = \frac{1}{\sigma^2} \sum_{p=1}^{n} \left[m_p - \frac{1}{2}(I + Q\,\cos 2\alpha_p + U\,\sin 2\alpha_p) \right]^2, \tag{6}$$

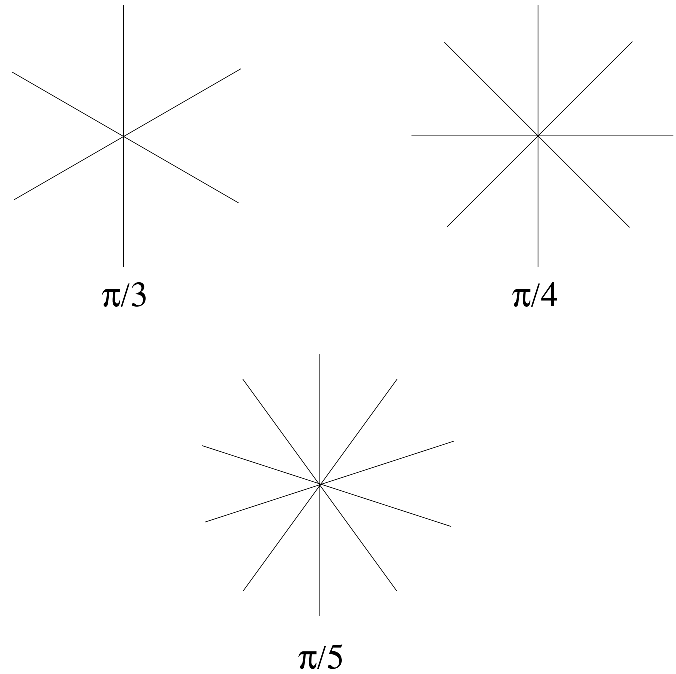

Fig. 1. The relative orientations of polarimeters in "Optimised Configurations" with 3, 4 and 5 detectors

and the inverse of the covariance matrix of the Stokes parameters is given by:

$$\mathbf{V}^{-1} = \frac{1}{\sigma^2}\mathbf{X}, \qquad \mathbf{X} = \mathbf{A}^T\mathbf{A} = \frac{1}{4}\ \times \tag{7}$$

$$\begin{pmatrix} n & \sum_1^n\,\cos 2\alpha_p & \sum_1^n\,\sin 2\alpha_p \\ \sum_1^n\,\cos 2\alpha_p & \frac{1}{2}(n + \sum_1^n\,\cos 4\alpha_p) & \frac{1}{2}\sum_1^n\,\sin 4\alpha_p \\ \sum_1^n\,\sin 2\alpha_p & \frac{1}{2}\sum_1^n\,\sin 4\alpha_p & \frac{1}{2}(n - \sum_1^n\,\cos 4\alpha_p) \end{pmatrix}.$$

It is shown in the appendix that, if the orientations of the polarimeters are evenly distributed on 180°:

$$\alpha_p = \alpha_1 + (p-1)\,\frac{\pi}{n},\ p = 1...\ n,\ \text{with } n \geq 3, \tag{8}$$

the matrix $\mathbf{V}$ takes the very simple diagonal form:

$$\mathbf{V}_0 = \sigma^2 {\mathbf{X}_0}^{-1},\ \text{with } {\mathbf{X}_0}^{-1} = \frac{4}{n} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{9}$$

independent of the orientation of the focal plane, while its determinant is minimised. In other words, at the same time, the errors on the Stokes parameters get decorrelated, their error matrix becomes independent of the orientation of the focal plane and the volume of the error ellipsoid takes its smallest possible value: $\frac{\pi}{3}\left(\frac{4\sigma}{\sqrt{n}}\right)^3$.

The "Optimised Configurations" (OC) are the sets of polarimeters which satisfy condition (8), (see Fig. 1). They are hereafter referred to by the subscript 0 as in Eq. (9). The smallest OC involves three polarimeters with relative angle $\pi/3$. With 4 polarimeters, the angular separation must be $\pi/4$, and so on. Note that a configuration with one unpolarised detector and 2 polarised detectors can never measure the Stokes parameters with uncorrelated errors, because this would require polarimeters oriented 90° apart from each other, which would not allow the breaking of the degeneracy between Q and U.

If one combines several OC's with several unpolarised detectors, all uncorrelated with each other, the resulting covariance matrix for the Stokes parameters remains diagonal and independent of the orientation of the various OC's. More precisely, when combining the measurements of n_T unpolarised detectors (temperature measurements), with n_P polarised detectors arranged in OC's, the covariance matrix of the Stokes parameters reads:

$$\mathbf{V} = \frac{4\,\sigma_P^2}{n_P} \begin{pmatrix} \left(1 + 4\,\frac{n_T}{n_P}\left(\frac{\sigma_P}{\sigma_T}\right)^2\right)^{-1} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{10}$$

where we have introduced inverse average noise levels, σ_T and σ_P, for the unpolarised and polarised detectors respectively:

$$\frac{1}{{\sigma_T}^2} = \left\langle \frac{1}{\sigma^2_{\text{unpolarised}}} \right\rangle, \text{ and } \frac{1}{{\sigma_P}^2} = \left\langle \frac{1}{\sigma^2_{\text{polarised}}} \right\rangle. \tag{11}$$

Note that the levels of noise can be different from one OC to the other and from those of the unpolarised detectors.

2.2. A more realistic description of the measurements

In general one expects that there will be some slight imbalance and cross-correlation between the noise of the detectors. The noise matrix of the measurements will in general take the form:

$$\mathbf{N} = \sigma^2(\mathbb{1} + \hat{\beta} + \hat{\gamma}), \tag{12}$$

where the imbalance $\hat{\beta}$ and cross-correlation $\hat{\gamma}$ matrices

$$\hat{\beta} = \begin{pmatrix} \beta_{1\,1} & 0 & 0 & \dots \\ 0 & \beta_{2\,2} & 0 & \dots \\ 0 & 0 & \beta_{3\,3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ \mathrm{Tr}(\hat{\beta}) = 0, \text{ and}$$

$$\hat{\gamma} = \begin{pmatrix} 0 & \gamma_{1\,2} & \gamma_{1\,3} & \dots \\ \gamma_{1\,2} & 0 & \gamma_{2\,3} & \dots \\ \gamma_{1\,3} & \gamma_{2\,3} & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{13}$$

are "small", that is will be treated as first order perturbations in the following, and therefore

$$\mathbf{N}^{-1} = \frac{1}{\sigma^2}\,(\mathbb{1} - \hat{\beta} - \hat{\gamma}).$$

To this order, the variance matrix of the Stokes parameters is easily obtained from Eq. (5):

$$\mathbf{V} = \sigma^2\left[\mathbf{X}^{-1} + \hat{\mathcal{B}} + \hat{\mathcal{G}}\right], \tag{14}$$

where the matrix $\mathbf{X}$ is given by Eq. (7) and the first order corrections to $\mathbf{V}$, matrices $\hat{\mathcal{B}}$ and $\hat{\mathcal{G}}$, read:

$$\begin{pmatrix} \hat{\mathcal{B}} \\ \hat{\mathcal{G}} \end{pmatrix} = \mathbf{X}^{-1}\ \mathbf{A}^T \begin{pmatrix} \hat{\beta} \\ \hat{\gamma} \end{pmatrix} \mathbf{A}\ \mathbf{X}^{-1}. \tag{15}$$

In an OC, the matrix $\mathbf{X}^{-1}$ takes the simple diagonal form ${\mathbf{X}_0}^{-1}$ of Eq. (9), and the non diagonal parts, $\hat{\mathcal{B}}$ and $\hat{\mathcal{G}}$ remain of order 1 in $\hat{\beta}$ and $\hat{\gamma}$. For instance, if we consider an OC with 3 polarimeters, and polarimeter number 1 is oriented in the x direction,

$$\hat{\mathcal{B}} = \frac{4}{3}\begin{pmatrix} 0 & \beta_{1\,1} & \frac{\beta_{2\,2}-\beta_{3\,3}}{\sqrt{3}} \\ \beta_{1\,1} & \beta_{1\,1} & \frac{\beta_{3\,3}-\beta_{2\,2}}{\sqrt{3}} \\ \frac{\beta_{2\,2}-\beta_{3\,3}}{\sqrt{3}} & \frac{\beta_{3\,3}-\beta_{2\,2}}{\sqrt{3}} & -\beta_{1\,1} \end{pmatrix},$$

where $\beta_{2\,2} + \beta_{3\,3} = -\beta_{1\,1}$, and

$$\hat{\mathcal{G}} = \frac{4}{3} \times \begin{pmatrix} \frac{2(\gamma_{1\,2}+\gamma_{1\,3}+\gamma_{2\,3})}{3} & \frac{\gamma_{1\,2}+\gamma_{1\,3}-2\gamma_{2\,3}}{3} & \frac{\gamma_{1\,2}-\gamma_{1\,3}}{\sqrt{3}} \\ \frac{\gamma_{1\,2}+\gamma_{1\,3}-2\gamma_{2\,3}}{3} & \frac{2(\gamma_{2\,3}-2\gamma_{1\,2}-2\gamma_{1\,3})}{3} & \frac{2(\gamma_{1\,2}-\gamma_{1\,3})}{\sqrt{3}} \\ \frac{\gamma_{1\,2}-\gamma_{1\,3}}{\sqrt{3}} & \frac{2(\gamma_{1\,2}-\gamma_{1\,3})}{\sqrt{3}} & -2\,\gamma_{2\,3} \end{pmatrix}.$$

Note that $\hat{\mathcal{B}}$ and $\hat{\mathcal{G}}$ transform under a rotation of the focal plane by a rotation $\mathbf{R}(\psi)$:

$$\begin{pmatrix} \hat{\mathcal{B}} \\ \hat{\mathcal{G}} \end{pmatrix} \to \mathbf{R}(\psi)^{-1}\begin{pmatrix} \hat{\mathcal{B}} \\ \hat{\mathcal{G}} \end{pmatrix}\mathbf{R}(\psi). \tag{16}$$

Because $\mathbf{V_0}$ is invariant, as long as $\hat{\beta}$ and $\hat{\gamma}$ are small the dependence of $\mathbf{V}$ on the orientation of the focal plane remains weak.

3. Co-adding measurements

The planned scanning strategy of Planck goes stepwise: at each step the satellite will spin about 100 times around a fixed axis, covering the same circular scan, then the spin axis of the satellite will be moved by a few arc-minutes, and so on. This provides two types of redundancy: every pixel along each circle will be scanned about 100 times, and some pixels will be seen by several circles, with different orientations of the focal plane. In this section we show, assuming a perfect white noise along each scan, that the properties of the error matrix of the Stokes parameters coming from OC's are kept if all data are simply co-added at each pixel, whatever the orientations of the focal plane. The redundancy provided by intersecting circles can be used to remove the stripes induced on maps by low-frequency noise in the data streams. An extension adapted to polarised measurements of the method proposed by Delabrouille (1998) for the de-striping of Planck maps is studied in Revenu et al. (1999).

Here we assume that the noise is not correlated between different scans and can thus be described by one matrix $\mathbf{N}_l$ for each scan, indexed by l, passing through the pixel. The χ^2 is then the sum of the χ_l^2 over the L scans that cross the pixel:

$$\chi^2 = \sum_{l=1}^{L} (\mathbf{M}_l - \mathbf{A}_l\,\mathbf{S})^T {\mathbf{N}_l}^{-1} (\mathbf{M}_l - \mathbf{A}_l\,\mathbf{S}). \tag{17}$$

The estimator of the Stokes parameters stemming from this χ^2 is

$$\mathbf{S} = \left(\sum_{l=1}^{L} \mathbf{A}_l^T\,\mathbf{N}_l^{-1}\,\mathbf{A}_l \right)^{-1} \sum_{l=1}^{L} \mathbf{A}_l^T\,\mathbf{N}_l^{-1}\,\mathbf{M}_l, \tag{18}$$

with variance matrix:

$$\mathbf{V} = \left(\sum_{l=1}^{L} \mathbf{A}_l^T\,\mathbf{N}_l^{-1}\,\mathbf{A}_l \right)^{-1}. \tag{19}$$

In the ideal case, for a given scan, the noise (assumed to be white on each scan) has the same variance for all bolometers with no correlation between them, although it can vary from one scan to the other:

$$\mathbf{N}_l = {\sigma_l}^2 \mathbb{1}, \tag{20}$$

and one can write the resulting variance combining the L scans:

$$\mathbf{V}_L = \left(\sum_{l=1}^{L} \frac{\mathbf{X}_l}{{\sigma_l}^2} \right)^{-1} = \left(\sum_{l=1}^{L} \frac{1}{{\sigma_l}^2} \mathbf{R}^{-1}(\psi_l)\,\mathbf{X}_1\,\mathbf{R}(\psi_l) \right)^{-1}, \tag{21}$$

where $\mathbf{X}_l = \mathbf{A}_l^T\,\mathbf{A}_l$, and we have written explicitly the rotation matrices which connect the orientation of the focal plane along scan l with that along scan 1. Note that these matrices are dependent of the position along the scan through angle ψ_l.

If the observing setup is in an OC, all orientation dependence drops out and the expression of the covariance matrix becomes diagonal as for a single measurement (Eq. 9):

$$\mathbf{V}_{0\,L} = \frac{4\,{\sigma_L}^2}{n\,L} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \frac{{\sigma_L}^2}{L}\,{\mathbf{X}_0}^{-1}, \tag{22}$$

where ${\mathbf{X}_0}^{-1}$ is defined in Eq. (9) and the average noise level σ_L is defined as:

$$\frac{1}{{\sigma_L}^2} = \left\langle \frac{1}{{\sigma_l}^2} \right\rangle. \tag{23}$$

Of course one recovers the fact that, with L measurements, the errors on the Stokes parameters are reduced by a factor $\sqrt{L}$.

More realistically, we expect that the noise matrices will take a form similar to Eq. (12):

$$\mathbf{N}_l = {\sigma_l}^2 \left(\mathbb{1} + \hat{\beta}_l + \hat{\gamma}_l \right). \tag{24}$$

If $\hat{\beta}_l$ and $\hat{\gamma}_l$ are small, first order inversion allows to calculate $\mathbf{V}$ ($\mathbf{V}_L$ is given by Eq. (21)):

$$\mathbf{V} = \mathbf{V}_L + \mathbf{V}_L \sum_{l=1}^{L} \mathbf{A}_l^T \frac{\hat{\beta}_l + \hat{\gamma}_l}{{\sigma_l}^2} \mathbf{A}_l \mathbf{V}_L. \tag{25}$$

If the focal plane is in an OC, this expression simplifies to

$$\mathbf{V}_L = \frac{{\sigma_L}^2}{L} \left({\mathbf{X}_0}^{-1} + \frac{{\sigma_L}^2}{L} \sum_{l=1}^{L} \frac{\hat{\mathcal{B}}_l + \hat{\mathcal{G}}_l}{{\sigma_l}^2} \right), \tag{26}$$

where

$$\begin{pmatrix} \hat{\mathcal{B}}_l \\ \hat{\mathcal{G}}_l \end{pmatrix} = \mathbf{R}^{-1}(\psi_l) {\mathbf{X}_0}^{-1}\,\mathbf{A}_1^T \begin{pmatrix} \hat{\beta}_l \\ \hat{\gamma}_l \end{pmatrix} \mathbf{A}_1\,{\mathbf{X}_0}^{-1} \mathbf{R}(\psi_l). \tag{27}$$

The $1/L$ factor inside the parenthesis in equation (26) implies that the cross-correlations and the dependence on on the orientation ψ_l of the focal plan remain weak when one cumulates measurements of the same pixel.

4. The error covariance matrix of the scalar E and B parameters

Scalar polarisation parameters, denoted E and B, have been introduced, which do not depend on the reference frame (Newman & Penrose 1966; Zaldarriaga & Seljak 1997). However, the properties of OCs do not propagate simply to the error matrix of the E and B parameters because their definition is non local in terms of the Stokes parameters.

Nevertheless, if the measurements errors are not correlated between different points of the sky (or if the correlation has been efficiently suppressed by the data treatment) then the elements of the error matrix of the multipolar coefficients $a_{E,lm}$ and $a_{B,lm}$ can be written in a simple form which is given in Appendix B for a general configuration.

For an OC, the error matrix simplifies further and its elements reduce to:

$$\begin{aligned} < \delta a_{{}^E_B,lm}\, \delta a_{{}^E_B,l'm'}{}^* > &= \frac{1}{2} \left(\frac{4\,\pi}{N_{pix}} \right)^2 \sum_{\hat{\mathbf{n}}_k} \sigma^2_{\rm Stokes}(\hat{\mathbf{n}}_k) \\ &\times \left[{}_2Y_{lm}(\hat{\mathbf{n}}_k)^*\, {}_2Y_{l'm'}(\hat{\mathbf{n}}_k) + {}_{-2}Y_{lm}(\hat{\mathbf{n}}_k)^*\, {}_{-2}Y_{l'm'}(\hat{\mathbf{n}}_k) \right] \\ < \delta a_{E,lm}\, \delta a_{B,l'm'}{}^* > &= \frac{i}{2} \left(\frac{4\,\pi}{N_{pix}} \right)^2 \sum_{\hat{\mathbf{n}}_k} \sigma^2_{\rm Stokes}(\hat{\mathbf{n}}_k) \\ &\times \left[{}_2Y_{lm}(\hat{\mathbf{n}}_k)^*\, {}_2Y_{l'm'}(\hat{\mathbf{n}}_k) - {}_{-2}Y_{lm}(\hat{\mathbf{n}}_k)^*\, {}_{-2}Y_{l'm'}(\hat{\mathbf{n}}_k) \right]. \end{aligned}$$

where N_{pix} is the total number of pixels in the sky, $\sigma_{\rm Stokes}$ is the common r.m.s. error on the Q and U Stokes parameters, $\hat{\mathbf{n}}_k$ is the direction of pixel k and functions ${}_{\pm 2}Y_{lm}(\hat{\mathbf{n}}_k)$ are the spin 2 spherical harmonics. If $\sigma_{\rm Stokes}$ does not depend on the direction in the sky, a highly improbable situation, then the orthonormality of the spin weighted spherical harmonics makes the error matrix fully diagonal in the limit of a large number of pixels:

$$\begin{pmatrix} < \delta a_{E,lm}\, \delta a_{E,l'm'}{}^* > & < \delta a_{E,lm}\, \delta a_{B,l'm'}{}^* > \\ < \delta a_{B,lm}\, \delta a_{E,l'm'}{}^* > & < \delta a_{B,lm}\, \delta a_{B,l'm'}{}^* > \end{pmatrix} = \mathbb{1}\, \frac{4\,\pi}{N_{pix}}\, \sigma^2_{\rm Stokes}\, \delta_{ll'} \delta_{mm'}.$$

Note that, even for unpolarised data, the error matrix between multipolar amplitude is not diagonal unless the r.m.s. error is constant over the whole sky (see for instance Oh, Spergel & Hinshaw 1998).

In the same conditions, the noise matrix of fields E and B is also fully diagonal:

$$\begin{pmatrix} \langle \delta E(\hat{\mathbf{n}})\ \delta E(\hat{\mathbf{n}}')\rangle & \langle \delta E(\hat{\mathbf{n}})\ \delta B(\hat{\mathbf{n}}')\rangle \\ \langle \delta B(\hat{\mathbf{n}})\ \delta E(\hat{\mathbf{n}}')\rangle & \langle \delta B(\hat{\mathbf{n}})\ \delta B(\hat{\mathbf{n}}')\rangle \end{pmatrix} = \mathbb{1}\ \sigma^2_{\rm Stokes}\ \delta_{\hat{\mathbf{n}}\,\hat{\mathbf{n}}'}.$$

5. Conclusion

In this paper we have shown that, if the noise of the polarimeters have nearly equal levels and are approximately uncorrelated, these detectors can be set up in “Optimised Configurations”. These configurations are optimised in two respects: first the volume of the error ellipsoid is minimised, and second the error matrix of the inferred Stokes parameters is approximately diagonal in all reference frames. This remains true even if one combines information from numerous measurements along different scanning circles. Such “Optimised Configurations”, with 3 and 4 polarimeters, have been proposed by the Planck High Frequency Instrument consortium (HFI Consortium, 1998) for the three channels where it is intended to measure the CMB polarisation.

Appendix A: The conditions for an OC

In this appendix, we show that conditions (8) diagonalise the error matrix $\mathbf{V}$ of the Stokes parameters and minimise its determinant, if the noises in the n polarimeters have identical levels and are not correlated.

We use the notation:

$$S_k = \sum_{p=1}^{n} \mathrm{e}^{i\,k\,\alpha_p} = |S_k|\,\mathrm{e}^{i\,\theta_k},\ \ k = 2, 4.$$

It can be seen from Eq. (7) that requiring that the error on I be decorrelated from the errors on Q and U is equivalent to the condition:

$$S_2 = 0. \tag{A1}$$

This condition can easily be fulfilled in a configuration where the angles $\alpha_{\rm p}$ are regularly distributed:

$$\alpha_p = \alpha_1 + (p-1)\,\delta\alpha,\ \ p = 1...\ n, \tag{A2}$$

with $n \geq 3,\ 0 < \delta\alpha < \pi,\ \delta\alpha \neq \pi/2$ (see text).

In such configurations, Eq. (A1) becomes:

$$S_2 = \mathrm{e}^{i\,2\,\alpha_1}\frac{\mathrm{e}^{i\,2\,n\,\delta\alpha} - 1}{\mathrm{e}^{i\,2\,\delta\alpha} - 1} = 0. \tag{A3}$$

The solutions of Eq. (A3) under conditions (A2) reduce to:

$$\delta\alpha = \frac{\pi}{n},\ \text{ with } n \geq 3. \tag{A4}$$

It is easily seen that conditions (A4) also automatically ensure that $S_4 = 0$ and therefore that $\mathbf{X}$ becomes diagonal and assumes the very simple form:

$$\mathbf{X}_0 = \frac{n}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \tag{A5}$$

independent of the orientation of the focal plane. Equation (9) is the consequence of (A5).

The error volume is proportional to the determinant of the error matrix $\mathbf{V}$. Therefore, it is minimum when the determinant of $\mathbf{X}$ (Eq. (7)) is maximum. This determinant can be written as:

$$\mathrm{Det}(\mathbf{X}) = \frac{1}{64}\left[n^3 - n\,|S_4|^2 - 2\,|S_2|^2(n - |S_4|\,\cos(\theta_4 - 2\,\theta_2))\right]. \tag{A6}$$

Because the S_k's are sums of n complex numbers with modulus 1, $|S_k| < n$, and it is clear from Eq. (A6) that

$$\mathrm{Det}(\mathbf{X}) \leq \frac{n^3}{64},$$

and that the upper bound is reached if and only if

$$S_2 = S_4 = 0. \tag{A7}$$

Conditions (A2) and (A4) have been shown above to imply (A7), and therefore ensure that the determinant of the covariance matrix $\mathbf{V}$ is minimum.

Appendix B: The general error matrix of the E and B multipolar coefficients

Assuming that the data treatment has removed all correlation between different directions in the sky, the matrix elements of the error matrix of the E and B multipolar coefficients are:

$$< \delta a_{{}^E_B,lm}\,\delta a_{{}^E_B,l'm'}{}^* >= \frac{1}{4}\left(\frac{4\,\pi}{N_{pix}}\right)^2 \sum_{\hat{\mathbf{n}}_k}$$
$$(\mathbf{N_{QQ}} + \mathbf{N_{UU}})(\hat{\mathbf{n}}_k)[{}_2Y_{lm}{}^*\,{}_2Y_{l'm'} + {}_{-2}Y_{lm}{}^*\,{}_{-2}Y_{l'm'}](\hat{\mathbf{n}}_k)$$
$$\pm\,(\mathbf{N_{QQ}} - \mathbf{N_{UU}})(\hat{\mathbf{n}}_k)[{}_2Y_{lm}{}^*\,{}_{-2}Y_{l'm'} + {}_{-2}Y_{lm}{}^*\,{}_2Y_{l'm'}](\hat{\mathbf{n}}_k)$$
$$\pm\,2i\,\mathbf{N_{QU}}(\hat{\mathbf{n}}_k)[{}_2Y_{lm}{}^*\,{}_{-2}Y_{l'm'} - {}_{-2}Y_{lm}{}^*\,{}_2Y_{l'm'}](\hat{\mathbf{n}}_k),$$

$$< \delta a_{E,lm}\,\delta a_{B,l'm'}{}^* >= \frac{i}{4}\left(\frac{4\,\pi}{N_{pix}}\right)^2 \sum_{\hat{\mathbf{n}}_k}$$
$$(\mathbf{N_{QQ}} + \mathbf{N_{UU}})(\hat{\mathbf{n}}_k)[{}_2Y_{lm}{}^*\,{}_2Y_{l'm'} - {}_{-2}Y_{lm}{}^*\,{}_{-2}Y_{l'm'}](\hat{\mathbf{n}}_k)$$
$$-\,(\mathbf{N_{QQ}} - \mathbf{N_{UU}})(\hat{\mathbf{n}}_k)[{}_2Y_{lm}{}^*\,{}_{-2}Y_{l'm'} - {}_{-2}Y_{lm}{}^*\,{}_2Y_{l'm'}](\hat{\mathbf{n}}_k)$$
$$-\,2i\,\mathbf{N_{QU}}(\hat{\mathbf{n}}_k)[{}_2Y_{lm}{}^*\,{}_{-2}Y_{l'm'} + {}_{-2}Y_{lm}{}^*\,{}_2Y_{l'm'}](\hat{\mathbf{n}}_k),$$

where $\mathbf{N}(\hat{\mathbf{n}}_\mathbf{k})$ is the noise matrix of the Stokes parameters Q and U in the direction $\hat{\mathbf{n}}_k$ of pixel k.

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References

Bond J.R., Efstathiou G., 1987, MNRAS 226, 655
Born M., Wolf E., 1980, Principles of Optics, sixth edition. Pergamon Press
Delabrouille J., 1998, A&AS 127, 555
HFI Consortium, 1998, High Frequency Instrument for the Planck Mission (submitted in response to the ESA Announcement of Opportunity)
Hu W., White M., 1997, New Astron. 2, 323
Melchiorri A., Vittorio N., 1996, proceedings of NATO Advanced Study Institute 1996 on the "Cosmic Background Radiation", astro-ph/961002
Newman E., Penrose R., 1966, J. Math. Phys. 7, 863
Ng K., Liu G., 1997, Correlation Functions of CMB Anisotropy and Polarization, astro-ph/9710012
Oh S.P., Spergel D.N., Hinshaw G., 1998, An Efficient Technique to Determine the Power Spectrum from Cosmic Microwave Background Sky Maps, astro-ph/9710012 (to appear in ApJ)
Rees M., 1968, ApJ 153, L1
Revenu B., Ansari R., Couchot F., Delabrouille J., Kaplan J., Kim A., 1999 (in preparation)
Seljak U., Zaldarriaga M., 1998, "Fundamental Parameters in Cosmology," proceedings of the XXXIIIrd Rencontres de Moriond 1998, Trân Thanh Vân et al. (ed.) Éditions Frontières, Paris, astro-ph/9805010
Zaldarriaga M., Seljak U., 1997, Phys. Rev. D 55, 1830