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# The geophysical approach towards the nutation of a rigid Earth

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**Abstract.** This paper presents the new and complete series named H96NUT for the nutation of a rigid Earth model. In contrast to the traditional computation method, the nutation is calculated here indirectly via the new and highly accurate tidal potential development recently given by Hartmann & Wenzel (1995a,b), (HW95). All contributions up to 0.45 $\mu$as were taken into account leading to 699 nutation terms separated after origin (or 607 terms summed according to the same arguments). These are: the main terms due to the Moon, the Sun and the planets, indirect planetary effects, effects due to the triaxiality of the Earth, effects due to the $J_3$ and $J_4$ geopotential, the geodesic nutation and second order terms. Nutation series for the angular momentum axis, the figure axis and the rotation axis are derived; they are in good agreement with those given in Roosbeek & Dehant (1997) with exception of several terms whose cause is discussed. Modern constants (like the precession constant) have been used throughout. The value for the dynamical ellipticity $H_{\rm dyn}$, which is compatible with these computations, is $H_{\rm dyn} = 3.273\,792\,489\;10^{-3}$. In addition, some other values related with precession are also deduced from the tidal potential. One main advantage of this work is that a completely independent analytical computation method has been established which is extensively described here (see also Hartmann (1996) for more details) and which provides – besides a numerical calculation – a useful validation of existing nutation series.

**Key words:** celestial mechanics, stellar dynamics — reference system — Earth

## 1. Introduction

The theory of the nutation for a rigid Earth model is of special interest. It still serves as a basis for the nutation of more complicated and realistic Earth models including elastic and anelastic effects of the mantle, effects from outer and inner core, effects from the oceans and the atmosphere etc. (see e.g. the ZMOA series by Herring 1991; or Mathews et al. 1991a,b; Dehant & Defraigne 1997; Herring 1996). Another important reason for improving the theoretical nutation series of a rigid Earth model is the increasing accuracy of modern space techniques such as Very Long Baseline Interferometry (VLBI) or the Global Positioning System (GPS). In particular with VLBI the nutation angles are observed with an accuracy of the order of 0.1 mas. Therefore present theoretical models for the nutation of a rigid Earth model (or briefly nutation models/series in the following) should deal with amplitudes much less than the observational accuracy limit in order not to contribute significantly to the total error budget. Note, that an accuracy of 1 $\mu$as corresponds to 0.031 mm on the Earth's surface.

For an overall threshold of a nutation series of about 1 $\mu$as, many different effects have to be taken into account. Although they depend somewhat on the computation method (e.g. the so-called second order terms, see below) one finds that the following contributions have to be included: the main effect (first and second order) from the Moon and the Sun, the direct and indirect influence of the planets (see e.g. Vondrák 1982, 1983a,b; Hartmann & Soffel 1994 and Souchay & Kinoshita 1997a, for the former) and the influence of some geophysical properties of the Earth like the spherical harmonics $J_3$, $J_4$ (see Hartmann et al. 1996) and the triaxiality coming due to the geopotential coefficients $C_{22}$ and $S_{22}$. In addition also a relativistic contribution called geodesic nutation (Fukushima 1991) has to be considered.

One of the first precise nutation theories has been set up by Woolard (1953). That series is fairly complete,

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although the truncation level is 200 $\mu$as only. It was followed by the theory of Kinoshita (1977), or briefly K77, with a threshold of 100 $\mu$as and 106 terms in total. This nutation model was modified for a deformable Earth by Wahr (1981) and is usually referred to as the standard IAU 1980 nutation series. Later, the rigid Earth nutation was improved by Zhu & Groten (1989), denoted by ZG89, and independently by Kinoshita & Souchay (1990), denoted by KS90, by one order of magnitude to 10 $\mu$as for ZG89 and 5 $\mu$as for KS90, so that they came up with 262 and 277 nutation terms, respectively. The detailed comparison and discussion between these two models can be found in Souchay (1993). Both series contain for the first time second order terms. Hartmann & Soffel (1994) and Williams (1995) treated a special part of the nutation model, namely the direct effect of the planets with thresholds of 2.5 $\mu$as and 0.5 $\mu$as, respectively.

New theories of the nutation for a rigid Earth model (containing all of the faint effects mentioned above) appeared in 1997, to be prepared for the General Assembly of IAU held in Kyoto, Japan at August 1997. There is an updated version of KS90's series, see Souchay & Kinoshita (1996, 1997a) where the corrections to the KS90 series are presented in parts due to some errors and change of the threshold to 0.5 $\mu$as but mainly due to an updated precession constant which is a scale factor for the nutation. Then Souchay & Kinoshita (1997b) extended their solution to the threshold 0.1 $\mu$as. That solution called SK96.2 by the authors contains nearly 1500 terms. The final version will be called REN-2000 and will be published in Souchay et al. (1998). The new theory of Roosbeek & Dehant (1997) denoted as RDAN97 with the same threshold 0.1 $\mu$as containing 1553 terms has appeared recently. This theory is based on the torque approach where the torque is derived from ephemerides in the frequency domain. Finally, Bretagnon et al. (1998) made an exhaustive analytical solution of the motion of the figure axis (and also of the rotation and angular momentum axes) of the Earth, based on the analytical theories of the motion of the Moon, the Sun and the planets of the Bureau des Longitudes, called SMART97 by the authors with the threshold of 0.01 $\mu$as and containing nearly 5000 terms that completes this list of nutation series.

The goal of this paper is to present a rigid Earth nutation theory called H96NUT computed independently and exclusively by a completely geophysical approach. As emphasized by Melchior & Georis (1968) the luni-solar torque, which is the driving force of nutation and precession, is directly related to the tesseral part of the tidal potential. This method of computation described therein was greatly extended and applied rigorously to derive a highly accurate nutation series. Since Hartmann & Wenzel (1995a,b) recently improved and drastically simplified the tidal potential development (called HW95) to a very high precision, all tools were at hand to start this investigation. Note that also ZG89 used this method partly, namely to compute their 156 additional terms to those 106 ones of the IAU nutation series. Obviously, this is not a homogeneous method of computation (as was already argued by Souchay 1993). As explained in Hartmann et al. (1996) all the terms in ZG89 due to $J_3$ are erroneous. Furthermore, the tidal potential used by ZG89 does not contain any planetary influence (in contrast to HW95 which contains both, the direct and the indirect planetary effects) and is inferior to HW95 by two orders of magnitude in truncation. All this provided another motivation to extend the geophysical approach.

The organization of this paper is as follows: in Sect. 2 the new nutation theory is presented. The computation of the tidal development is briefly described there, the different coordinate systems are discussed, the relationship between the torque and the tidal potential is given, the nutation theory is developed in time domain using a perturbation theory up to second order and some principle aspects of the new computation method are discussed. Then, in Sect. 3 some of the results of this procedure are given. Section 4 tries to give some error estimates for the individual nutation terms. Then, a comparison with the theories of other authors is given in Sect. 5. Finally, some conclusions are drawn in Sect. 6.

## 2. Theory

### *2.1. Some words about the principal methods of computation of the nutation*

Let us recall some aspects of the general strategy of how to compute the orientation of the Earth in space. Basically nutation, together with precession, represents the motion of the Earth's axis with respect to an inertial frame. To be more precise, the Earth as a member of the solar system undergoes a translational motion usually described by the motion of its center of mass in space and a rotational motion around the center of mass (spin, nutation, precession). The precession and nutation have the same origin, namely the gravitational attraction between the Earth's equatorial bulge and the other bodies. For the rotational motion that force acts on the spinning Earth and makes it moving like a gyro under the influence of a torque. To describe its rotational motion theoretically we can find three possibilities.

1) First, employing the "classic" computation method by using Lagrange theory and a force function. Woolard's classical treatment (1953) was formulated in frames of Lagrange theory. Taking the Euler angles $\theta, \psi$ and $\varphi$ as generalized coordinates $q_i, i = 1,2,3$, the Lagrangian reads

$$\mathcal{L} = U + \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2), \qquad (1)$$

where $A, B$ and $C$ are the principle moments of inertia and $U$ is the so-called force function for the external forces.

Relating the components $\omega_i$ of the rotation vector of the principal axes system of the Earth with the Euler angles using Euler's kinematical equations (shortly EKE), we find:

$$\begin{aligned} \omega_1 &= \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\ \omega_2 &= \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi \\ \omega_3 &= \dot{\varphi} + \dot{\psi}\cos\theta. \end{aligned} \tag{2}$$

Next, we have Lagrange's equations for a conservative system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} - \frac{\partial\mathcal{L}}{\partial q_i} = 0. \tag{3}$$

Then the Euler's dynamical equations (shortly EDE) are:

$$\begin{aligned} A\dot{\omega}_1 + (C-B)\omega_2\omega_3 &= \frac{\sin\varphi}{\sin\theta}\left(\cos\theta\frac{\partial U}{\partial\varphi} - \frac{\partial U}{\partial\psi}\right) - \cos\varphi\frac{\partial U}{\partial\theta} \\ B\dot{\omega}_2 + (A-C)\omega_3\omega_1 &= \frac{\cos\varphi}{\sin\theta}\left(\cos\theta\frac{\partial U}{\partial\varphi} - \frac{\partial U}{\partial\psi}\right) + \sin\varphi\frac{\partial U}{\partial\theta} \\ C\dot{\omega}_3 + (B-A)\omega_1\omega_2 &= \frac{\partial U}{\partial\varphi}. \end{aligned} \tag{4}$$

Integrating these differential equations yields the expressions that are required for the theory of the motion of the Earth relative to its center of mass. This is – together with the calculation of the force function $U$ for the gravitational action between the Earth and an external massive gravitating body – the main task for this computation method. Analytical ephemerides are (almost) exclusively used for the determination of the positions of the external body from which $U$ can be developed. Depending on the coordinate systems involved (especially those used for the ephemerides), the computation of $U$ can be – despite of all symbol manipulating programs – quite laborious. Also solving the differential Eqs. (4) analytically is not an easy task.

2) Another method is based on the Hamiltonian formalism. This approach was introduced by Andoyer (1923, 1926) and perfected in Kinoshita (1977), Kinoshita & Souchay (1990) and Souchay & Kinoshita (1996, 1997a). Kinoshita has developed the analytical theory describing the motion of the planes normal to the angular momentum vector, to the figure axis and to the rotation axis of the rigid Earth having the shape of a triaxial ellipsoid. The Hamiltonian formalism which was used is characterized by the use of Andoyer variables, a moving fundamental reference frame and Hori's perturbation method. The force function $U$ is to be based on the analytical ephemerides, likewise in the case of using of the Lagrange theory.

Although the force function $U$ can be considered as a small quantity thus allowing for a perturbative treatment, some special attention has to be given to the separation of the precession (long-periodic contribution) and the nutation (short-periodic contribution). Kinoshita (1972), K77 and Kinoshita & Souchay (1990) describe in detail how they use canonical transformations to tackle this problem.

3) The other approach was used to compute the nutation model H96NUT presented here. It differs from the classical method mainly in three aspects. First, instead of the force function $U$ the external torque $\boldsymbol{T}$ is used directly and expressed entirely in terms of tidal quantities which can be taken from the tidal potential development HW95. Roosbeek & Dehant (1997) used also the external torque but they developed it from the ephemerides of the perturbing bodies. Second, the differential equations are solved directly (using also a perturbation method) in the time domain. Third, the coordinate system used for most of the calculations is the system of principal axes of the Earth. The transformation to the ecliptic of date is applied only in a final stage. All of this is described in the following sections. More detailed derivations of the formulas can be found in Hartmann (1996).

### *2.2. The tidal potential development HW95*

The first point to be discussed is the computation of the tidal potential, which replaces the rather complicated calculation of the disturbing function $U$ in K77 or KS90. Here, only a summary of the most important aspects is given. More details can be found in Hartmann & Wenzel (1995a,b), where a time-harmonic development of the Earth tide generating potential (tidal potential) called HW95 due to the Moon, the Sun and the planets is presented. It contains 12 935 tidal waves and is the most accurate tidal potential series available today (see Wenzel 1996). The expansion is carried out to the degree 6 of the geopotential for the Moon, degree 3 for the Sun and degree 2 for the planets Mercury, Venus, Mars, Jupiter and Saturn. It is entirely based on the JPL standard numerical ephemerides DE200. The secular variation of the biggest amplitudes is computed for the period between 1850 and 2150. The indirect effects of the planets, mainly due to Venus and Jupiter, on the Earth's orbit and the Moon's orbit are taken into account implicitly by the use of the numerical ephemerides. This also applies to the so-called $J_2$-tilt effect, where the $J_2$ coefficient of the Earth causes the orbit of the Moon to be slightly tilted.

The tidal potential $V(t)$ due to a specific body (labeled $b$) at the surface of the Earth is given by

$$\begin{aligned} V(t) &= GM_\mathrm{b}\sum_{\ell=2}^{\ell_\mathrm{max}}\sum_{m=0}^{\ell}\frac{r^\ell}{r_\mathrm{b}^{\ell+1}(t)}\frac{1}{2\ell+1}\times \\ &\quad\times\overline{P}_{\ell m}(\cos\theta)\overline{P}_{\ell m}(\cos\theta_\mathrm{b}(t))\cos[m(\lambda-\Lambda_\mathrm{b}(t))]. \end{aligned} \tag{5}$$

Here, $r$, $\theta$, $\lambda$ denote the geocentric spherical coordinates of the station on (or near) the Earth's surface where $r$ is the radial distance, $\theta$ is the co-latitude and $\lambda$ is the longitude and $r_\mathrm{b}$, $\theta_\mathrm{b}$, $\Lambda_\mathrm{b}$ are those of the center of mass of the specific planet, whose mass is denoted by $M_\mathrm{b}$. $G$ is the gravitational constant. The fully normalized Legendre functions of degree $\ell$ and order $m$ used for the computation of the catalogue of the tidal potential HW95 are denoted by $\overline{P}_{\ell m}$. Then, the catalogue HW95 has been computed by using an expansion of the form

$$V(t) = \sum_{\ell=2}^{\ell_{\max}} \sum_{m=0}^{\ell} \left(\frac{r}{a}\right)^{\ell} \overline{P}_{\ell m}(\cos\theta) \times \times \sum_{i} \left[C_i^{\ell m}(T)\cos(\alpha_i(T)) + S_i^{\ell m}(T)\sin(\alpha_i(T))\right] \quad (6)$$

where $a$ is the major semi axis of the Earth and $T$ is the time reckoned from J2000 in Julian centuries. The time dependent tidal potential coefficients (shortly the tidal amplitudes) are given by

$$C_i^{\ell m}(T) = \mathrm{C0}_i^{\ell m} + T \cdot \mathrm{C1}_i^{\ell m}, \quad (7)$$

$$S_i^{\ell m}(T) = \mathrm{S0}_i^{\ell m} + T \cdot \mathrm{S1}_i^{\ell m}, \quad (8)$$

where the potential coefficients $\mathrm{C0}_i^{\ell m}$ and $\mathrm{S0}_i^{\ell m}$ have the dimension $\mathrm{m}^2\ \mathrm{s}^{-2}$ and the linear drift coefficients $\mathrm{C1}_i^{\ell m}$ and $\mathrm{S1}_i^{\ell m}$ have the dimension $\mathrm{m}^2\ \mathrm{s}^{-2}\ \mathrm{cy}^{-1}$. They are – together with the tidal arguments $\alpha_i(T)$ – the fundamental quantities in this paper from which everything else is deduced. The tidal arguments $\alpha_i(T)$ are computed from

$$\alpha_i(T) = m \cdot \lambda + \sum_{j=1}^{11} k_{ij} \cdot \arg_j(T). \quad (9)$$

The integer coefficients $k_{ij}$ are given in the HW95 catalogue, while the eleven astronomical arguments $\arg_j(T)$ ($j = 1, \ldots, 11 : \tau = t + h - s - \lambda$ is the mean local lunar time and $t$ is mean solar time, $s =$ mean lunar longitude, $h =$ mean solar longitude, $p =$ mean longitude of lunar perigee, $N' =$ negative mean longitude of the lunar ascending node, $p_{\mathrm{s}} =$ mean longitude of solar perigee and the mean longitudes of Mercury, Venus, Mars, Jupiter and Saturn indicated by $l_{\mathrm{Me}}$, $l_{\mathrm{V}}$, $l_{\mathrm{Ma}}$, $l_{\mathrm{J}}$ and $l_{\mathrm{S}}$, respectively which are referred to the mean dynamical ecliptic and equinox of date) are computed from polynomials in time, that can be found in Simon et al. (1994). This set of arguments is also the basis for the nutation, but with $s, h, p, N'$ and $p_{\mathrm{s}}$ replaced by $l_{\mathrm{m}}, l_{\mathrm{s}}, F, D$ and $\Omega$ using the well-known linear relations: $l_{\mathrm{m}} = s - p,\ l_{\mathrm{s}} = h - p_{\mathrm{s}},\ F = s + N',\ D = s - h$ and $\Omega = -N'$. For the nutation the tidal arguments may be considered as linear functions of time. Corrections to this approximation are discussed in the Sect. 3.5.

The numerical standard ephemerides DE200 (Standish & Williams 1981) supplied by Jet Propulsion Laboratory, Pasadena, have been used to compute the celestial geocentric vector $\boldsymbol{r}_{\mathrm{cel.}}$ for the time interval 1850 to 2150. The conversion to terrestrial geocentric vector $\boldsymbol{r}_{\mathrm{terr.}}$ uses the well-known relation

$$\boldsymbol{r}_{\mathrm{terr.}} = \mathcal{S} \cdot \mathcal{N} \cdot \mathcal{P} \cdot \boldsymbol{r}_{\mathrm{cel.}}. \quad (10)$$

The mean obliquity, the precession constant and the precession formulas of Simon et al. (1994) have been used for $\mathcal{P}$ including corrections for the new values of planetary masses. The IAU 1980 Nutation theory (Seidelmann 1982) has been used for $\mathcal{N}$ with the Delaunay arguments replaced by those of Simon et al. (1994). To assure the precision of the tidal potential of $10^{-8}\ \mathrm{m}^2\ \mathrm{s}^{-2}$ we need the coordinates of the attracting bodies with an accuracy better than 30 mas. Taking into account the known difference of the IAU 1980 Nutation theory and recent observations which are less then 10 mas we are justified to use the IAU 1980 Nutation theory. The model of Earth used in the computation for $\mathcal{S}$ rotates according to the expression of Aoki et al. (1982, Eq. 14). The equation of equinoxes was added to obtain the Greenwich Apparent Sidereal Time (GAST)

$$\mathrm{GAST} = \mathrm{GMST} + \Delta\Psi \cdot \cos\varepsilon. \quad (11)$$

All numerical values for the astronomical constants were taken from the IERS 1992 Standards (McCarthy 1992).

It is important to note that the transformation from (5) to (6) requires the computation of a time series of the tidal potential (about 300 years were used) and a spectral analysis to find the individual terms in the expansion (6) see Hartmann (1996) and Hartmann & Wenzel (1995a,b) for details. In addition, the integer numbers $k_{ij}$ for the tidal and nutational argument had to be found from the particular value of the interpolated Fourier-frequency. The numerical separation of waves with very small frequency differences (e.g. $2p_{\mathrm{s}}$ corresponding to a period of 10 468 years which also occurs as a difference of two nutation arguments) as well as the final total fit of the tidal amplitudes caused some problems that will appear later also in the corresponding nutation series (see also Roosbeek 1996).

### *2.3. The relationship between the torque and the tidal potential*

Next, the relationship between the external torque $\boldsymbol{T}$ and the HW95 tidal potential quantities will be calculated. In Hartmann et al. (1994) it is shown that this formula can be derived without integration via symmetric, trace-free tidal moments. Using again the system based on the principal axes of the Earth we find for the main contribution of the tidal potential of degree $\ell = 2$ (omitting the sum $i$ over all tides whenever possible) the exact expression

$$\boldsymbol{T} = \frac{\sqrt{15}}{a^2} \begin{pmatrix} (C-B)\,[S_i^{21}\cos(\alpha_i(T)) - C_i^{21}\sin(\alpha_i(T))] \\ (A-C)\,[C_i^{21}\cos(\alpha_i(T)) + S_i^{21}\sin(\alpha_i(T))] \\ (B-A)\,[S_i^{22}\cos(\alpha_i(T)) + C_i^{22}\sin(\alpha_i(T))] \end{pmatrix}, \quad (12)$$

where $a$ is taken to be the semi-major axis of the Earth from IERS 1992 Standards (McCarthy 1992). For an axisymmetric Earth ($A = B$) only the tesseral part of the tidal potential (order $m = 1$) is involved, but this is not assumed in the following in order to obtain the triaxiality terms. The sectorial part ($m = 2$) slightly influences the length of day as shown in e.g. Wünsch (1991) but this can be ignored.

Similar relations can be found for the higher parts ($\ell = 3, 4, \ldots$) of the tidal potential (5). For the nutation coming from these parts the higher zonal mass multipole moments ($J_3, J_4$) are the dominant contributions to the torque. Then, the torque can again be expressed (now entirely) in terms of the tesseral part ($m = 1$) of the tidal potential. Due to the advantageous representation (6) of the HW95 tidal potential, these contributions to the nutation can be obtained by replacing the scale factor and the tidal amplitudes (see Hartmann et al. 1996; for more details). To simplify the following calculations, the factor $\sqrt{15}/a^2$ is absorbed into the tidal amplitudes.

### *2.4. The free rotational motion of the Earth*

The rotational equation of motion in the figure axes system, in which the torque is given by (12), reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{S} + \boldsymbol{\omega} \times \boldsymbol{S} = \boldsymbol{T}, \tag{13}$$

where the torque $\boldsymbol{T}$ is included for later purposes; it vanishes for free motion. The spin (or angular momentum) vector $\boldsymbol{S}$ of the Earth is related to the rotation vector $\boldsymbol{\omega}$ by the moment of inertia tensor $\mathsf{C}$, which – again in the figure axis system – reads

$$\boldsymbol{S} = \mathsf{C} \cdot \boldsymbol{\omega} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \cdot \boldsymbol{\omega}. \tag{14}$$

Written out in components this leads again to EDE given in (4) and which therefore relate the components of the rotation vector $\boldsymbol{\omega}$ in the figure axes system to those of the torque, or with the help of the result of the last section, to the tidal quantities. The exact solution of EDE with vanishing torque can be deduced by means of elliptical functions. However, their modulus $m$

$$m = \frac{\omega_{10}^2}{\omega_{30}^2}\frac{A(B-A)}{C(C-B)} \tag{15}$$

is so small for the Earth ($\approx 10^{-14}$) that they can be replaced by their trigonometrical counterparts. Therefore we proceed with

$$\begin{aligned} \omega_1(t) &\approx \omega_{10}\cos(\mu\,(t-t_0)) \\ \omega_2(t) &\approx \omega_{20}\sin(\mu\,(t-t_0)) \\ \omega_3(t) &\approx \omega_{30}. \end{aligned} \tag{16}$$

The integration constants are $\omega_{10}, \omega_{30}$ and $t_0$. The two other constants are defined by

$$\omega_{20}^2 \equiv \omega_{10}^2 \frac{A}{B}\frac{C-B}{C-A} \quad \text{and} \quad \mu^2 \equiv \frac{C-A}{A}\frac{C-B}{B}\,\omega_{30}^2. \tag{17}$$

For the Earth the ratios $\omega_{10}/\omega_{30} \approx \omega_{20}/\omega_{30}$ describing the angle between the rotation axis and the figure axis are a few $0.''1 \approx 0.5\ 10^{-6}$ rad. Finally EKE in Eq. (2), must be solved with the left hand side being given by (16). The solution of the rearranged EKE

$$\begin{aligned} \dot\psi\sin\theta &= \sin\varphi\,\omega_1(t) + \cos\varphi\,\omega_2(t) \\ \dot\theta &= \cos\varphi\,\omega_1(t) - \sin\varphi\,\omega_2(t) \\ \dot\varphi &= \omega_3(t) - \dot\psi\cos\theta \end{aligned} \tag{18}$$

can be obtained by integration and gives with good approximation:

$$\begin{aligned} \psi(t) &= \psi_0 - \frac{A_+}{\sin\theta_0}\cos\varphi_+ - \frac{A_-}{\sin\theta_0}\cos\varphi_- \\ \theta(t) &= \theta_0 + A_+\sin\varphi_+ + A_-\sin\varphi_- \\ \varphi(t) &= \varphi_0 + \omega_{30}t + \cot\theta_0\{A_+\cos\varphi_+ + A_-\cos\varphi_-\} \end{aligned} \tag{19}$$

with $\varphi_\pm \equiv \varphi_0 + [\omega_{30} \pm \mu]\,t$ and constants

$$A_+ \equiv \frac{\omega_{10}+\omega_{20}}{2\,(\omega_{30}+\mu)} \approx \quad 1.5\ 10^{-6}\,\text{rad} \tag{20}$$

$$A_- \equiv \frac{\omega_{10}-\omega_{20}}{2\,(\omega_{30}-\mu)} \approx -2.4\ 10^{-9}\,\text{rad}. \tag{21}$$

$\psi_0, \theta_0$ and $\varphi_0$ are three integration constants. Neglected terms contribute with less than 1 $\mu$as or 5 $10^{-12}$ rad. Note that the result (19) does not give the usual nutation angles but the Euler angles between the figure axis system and an inertial system. How to extract the nutation angles from them will be explained in the Sect. 2.7.

### *2.5. Perturbation theory for the forced motion of the Earth*

#### 2.5.1. Solution of EDE with external torque

Now, the external luni-solar and planetary torques are included. It turns out that in order to satisfy the first two EDE's, (4), to our level of accuracy $\omega_3(t) = \omega_{30} = \text{const}$ is sufficient. Then, by using the substitutions

$$\begin{aligned} \omega_1(t) &= \omega_1^{\text{free}} + (\omega_{1\text{c}} + t\,\omega_{1\text{ct}})\cos(\phi_i + \omega_i\,t) + \\ &\quad + (\omega_{1\text{s}} + t\,\omega_{1\text{st}})\sin(\phi_i + \omega_i\,t) \end{aligned} \tag{22}$$

$$\begin{aligned} \omega_2(t) &= \omega_2^{\text{free}} + (\omega_{2\text{c}} + t\,\omega_{2\text{ct}})\cos(\phi_i + \omega_i\,t) + \\ &\quad + (\omega_{2\text{s}} + t\,\omega_{2\text{st}})\sin(\phi_i + \omega_i\,t) \end{aligned} \tag{23}$$

$$\omega_3(t) = \omega_{30}, \tag{24}$$

where $\omega_1^{\text{free}}, \omega_2^{\text{free}}$ are the free solutions according to (16) describing the Eulerian (= torque-free) motion of the Earth, one can satisfy the EDE by an appropriate choice of the eight constant coefficients $\omega_{1\text{c}}, \omega_{1\text{ct}}, \ \ldots$ They depend on the tidal amplitudes $C_i^{21}, S_i^{21}$, tidal phases $\phi_i$ and frequencies $\omega_i$ and on the constants $\omega_{10}, \omega_{20}, \omega_{30}, \mu$ introduced in the last section. Further corrections on $\omega_3$ are not necessary for the computation of the nutations since they are smaller than $\omega_{30}$ by a relative factor of at least $10^{-9}$.

#### 2.5.2. Solution of EKE with external torque

To solve EKE in the form (18), we have to substitute the solution (22–24) into their right hand sides. Rewriting the products of the trigonometric functions as sums and subtractions of their arguments, we obtain

$$\begin{aligned} \sin\theta\,\dot\psi = \; & \bar a_3 + \bar a_4\,t + \frac{\omega_{10}-\omega_{20}}{2}\sin(\varphi-\mu) + \\ & + \frac{\omega_{10}+\omega_{20}}{2}\sin(\varphi+\mu) + \\ & + (a_1 + a_2\,t)\sin(\varphi - \phi_i - \omega_i\,t) + \\ & + (a_3 + a_4\,t)\cos(\varphi - \phi_i - \omega_i\,t) + \\ & + (a_5 + a_6\,t)\sin(\varphi + \phi_i + \omega_i\,t) + \\ & + (a_7 + a_8\,t)\cos(\varphi + \phi_i + \omega_i\,t) \end{aligned} \tag{25}$$

$$
\begin{aligned}
\dot\theta = \; & \bar b_3 + \bar b_4\, t + \frac{\omega_{10}-\omega_{20}}{2}\cos(\varphi-\mu) + \\
& + \frac{\omega_{10}+\omega_{20}}{2}\cos(\varphi+\mu) + \\
& + (b_1 + b_2\, t)\sin(\varphi - \phi_i - \omega_i\, t) + \\
& + (b_3 + b_4\, t)\cos(\varphi - \phi_i - \omega_i\, t) + \\
& + (b_5 + b_6\, t)\sin(\varphi + \phi_i + \omega_i\, t) + \\
& + (b_7 + b_8\, t)\cos(\varphi + \phi_i + \omega_i\, t) +
\end{aligned} \tag{26}
$$

$$
\dot\varphi = \omega_{30} - \cos\theta\,\dot\psi \tag{27}
$$

with new constants $a_{1,\,...,\,8}$ and $b_{1,\,...,\,8}$ simply given as sums and differences of the former $\omega_{1c}$, $\omega_{1ct}$, ... It turns out that the constants $a_3 = -b_1 = (\omega_{1s} + \omega_{2c})/2$ are the largest. The constant terms $\bar a_3, \bar a_4, \bar b_3$ and $\bar b_4$ denote the contributions from the $K_1$-tide for which $\varphi - \phi_i - \omega_i t$ vanishes and have to be separated from the other tides in the fourth lines since they determine the precession. The terms including $\omega_{10} \pm \omega_{20}$ result from the torque-free motion. Terms including arguments $(\varphi - \phi_i - \omega_i\, t)$ will give the "real" in- and out-of-phase nutation terms, while those including $(\varphi + \phi_i + \omega_i\, t)$ will give the terms coming from the triaxiality of the Earth (also in- and out-of-phase, but the latter are negligible). The task is now to solve these differential equations by means of successive approximations.

#### 2.5.3. Solution of EKE with external torque at the first order

In a first approximation, we neglect the second term in (27) and put $\sin\theta$ equal to $\sin\theta_0$ in (25), where $\theta_0$ is the constant mean obliquity at J2000. Then, by integration we find immediately:

$$
\varphi_1 = \varphi_0 + \omega_{30}\, t \tag{28}
$$

$$
\begin{aligned}
\sin\theta_0\,\psi_1 = \; & \psi_0 + \bar a_3 t + \frac{\bar a_4}{2}t^2 - \\
& - A_+\cos(\varphi_0 + [\omega_{30}+\mu]\, t) - \\
& - A_-\cos(\varphi_0 + [\omega_{30}-\mu]\, t) + \\
& + \sin(\Delta\phi_i + \Delta\omega_i\, t)\left[\frac{a_3}{\Delta\omega_i} - \frac{a_2}{\Delta\omega_i^2} + \frac{a_4}{\Delta\omega_i}\, t\right] + \\
& + \cos(\Delta\phi_i + \Delta\omega_i\, t)\left[\frac{a_1}{\Delta\omega_i} + \frac{a_4}{\Delta\omega_i^2} + \frac{a_2}{\Delta\omega_i}\, t\right] + \\
& + \sin(\Sigma\phi_i + \Sigma\omega_i\, t)\left[\frac{a_7}{\Sigma\omega_i} + \frac{a_6}{\Sigma\omega_i^2} + \frac{a_8}{\Sigma\omega_i}\, t\right] + \\
& + \cos(\Sigma\phi_i + \Sigma\omega_i\, t)\left[-\frac{a_5}{\Sigma\omega_i} + \frac{a_8}{\Sigma\omega_i^2} - \frac{a_6}{\Sigma\omega_i}\, t\right]
\end{aligned} \tag{29}
$$

$$
\begin{aligned}
\theta_1 = \; & \theta_0 + \bar b_3 t + \frac{\bar b_4}{2}t^2 + \\
& + A_+\sin(\varphi_0 + [\omega_{30}+\mu]\, t) + \\
& + A_-\sin(\varphi_0 + [\omega_{30}-\mu]\, t) + \\
& + \sin(\Delta\phi_i + \Delta\omega_i\, t)\left[\frac{b_3}{\Delta\omega_i} - \frac{b_2}{\Delta\omega_i^2} + \frac{b_4}{\Delta\omega_i}\, t\right] + \\
& + \cos(\Delta\phi_i + \Delta\omega_i\, t)\left[\frac{b_1}{\Delta\omega_i} + \frac{b_4}{\Delta\omega_i^2} + \frac{b_2}{\Delta\omega_i}\, t\right] + \\
& + \sin(\Sigma\phi_i + \Sigma\omega_i\, t)\left[\frac{b_7}{\Sigma\omega_i} + \frac{b_6}{\Sigma\omega_i^2} + \frac{b_8}{\Sigma\omega_i}\, t\right] + \\
& + \cos(\Sigma\phi_i + \Sigma\omega_i\, t)\left[-\frac{b_5}{\Sigma\omega_i} + \frac{b_8}{\Sigma\omega_i^2} - \frac{b_6}{\Sigma\omega_i}\, t\right],
\end{aligned} \tag{30}
$$

with the abbreviations

$$
\begin{aligned}
\Delta\phi_i &= \phi_i - \varphi_0, \qquad & \Sigma\phi_i &= \phi_i + \varphi_0, \\
\Delta\omega_i &= \omega_i - \omega_{30}, \qquad & \Sigma\omega_i &= \omega_i + \omega_{30}.
\end{aligned} \tag{31}
$$

The interpretation of the various terms is quite obvious. $\varphi_1$ describes (at the first order) the Earth's rotation around the figure axis. The first lines of (29) and (30) are secular contributions describing the precession. Next comes the contribution from the torque-free motion (the terms containing $A_+$ and $A_-$). Finally, the in- and out-of-phase short-periodic terms with periods approximately between 4 days and 18.6 years and the in- and out-of-phase terms coming from the triaxiality of the Earth with periods of about half a day occur. It should be mentioned that this solution also gives the Euler angles and not the nutation angles themselves (see below).

#### 2.5.4. Solution of EKE with external torque at the second order

Now second order corrections necessary to obtain the desired accuracy will be computed. They need to be applied only to the first order terms being large enough, say 10 mas, i.e. to the precession and the in-phase nutation, but not to the free motion, the triaxiality terms and the out-of-phase nutation terms.

First, the correction for the angle $\varphi$ due to the neglected second term in (27) can be computed from

$$
\dot\varphi_2 = -\cos\theta\dot\psi \tag{32}
$$

and leads to (taking into account for $\psi$ only the precession $\bar a_3$ and the nutation $a_{3i}$ in longitude)

$$
\begin{aligned}
\varphi_2 \;&\approx\; -\cos\theta_0\,\psi_1 \approx \\
&\approx\; -\cot\theta_0\left[\bar a_3\, t + \frac{a_{3i}}{\Delta\omega_i}\sin(\Delta\phi_i + \Delta\omega_i\, t)\right].
\end{aligned} \tag{33}
$$

From now a subscript $i$ on the constants $a_3, b_1, \ldots$ is written explicitly to show that they depend on the $i$-th tide. At the first degree of approximation we get: $a_{3i} = -b_{1i} \approx (S_i^{21}/\omega_i)H_{\rm dyn}$, where $S_i^{21}$ is the tidal amplitude and $\omega_i$ the tidal frequency. An interesting feature of (33) is that the first contribution to $\varphi_2$, which is nothing else but the precession in right ascension, can be included into a modified Earth rotation speed $\tilde\omega_{30} = \omega_{30} - \cot\theta_0\bar a_3$. In other words, $\omega_{30}$ and $\tilde\omega_{30}$ stand for the Earth's rotation speed in a precessing and inertial coordinate system, respectively. The other contribution in (33) is simply the first order difference between GMST and GAST which is the "equation of equinoxes". Since it is a periodic contribution with a small amplitude an expansion only at the first order is possible.

The second order correction $\theta_2$ in obliquity is then computed. Again EKE have to be solved but now with $\varphi = \varphi_0 + \tilde{\omega}_{30}\, t + (a_{3i}/\Delta\omega_i)\sin(\Delta\phi_i + \Delta\omega_i\, t)$ instead of $\varphi = \varphi_0 + \omega_{30}\, t$ in the arguments on the right hand side of (26). First, we get the solution (30) with $\omega_{30}$ being replaced by $\tilde{\omega}_{30}$ (which also transforms $\Delta\omega_i = \omega_i - \omega_{30}$ into $\Delta\tilde{\omega}_i = \omega_i - \tilde{\omega}_{30}$). Although this modified solution may be considered to contain "second order" contributions, we still call these terms first order terms here. Second, we get a true second order term from the expansion with respect to the periodic correction in $\varphi_2$. By integrating

$$\dot{\theta}_2 \;=\; \sum_{i,j} - \cot\theta_0 \frac{a_{3j}}{\Delta\tilde{\omega}_j} b_{1i} \times \times \cos(\Delta\phi_i + \Delta\tilde{\omega}_i\, t)\sin(\Delta\phi_j + \Delta\tilde{\omega}_j\, t) \tag{34}$$

and after some rearranging and using $b_{1i} = -a_{3i}$, we obtain

$$\theta_2 \;=\; \sum_{i,j} \frac{\cot\theta_0}{4} \frac{a_{3i}}{\Delta\tilde{\omega}_i} \frac{a_{3j}}{\Delta\tilde{\omega}_j} \times \times\Big\{ - \cos(\Delta\phi_i - \Delta\phi_j + [\Delta\tilde{\omega}_i - \Delta\tilde{\omega}_j]\, t) + + \cos(\Delta\phi_i + \Delta\phi_j + [\Delta\tilde{\omega}_i + \Delta\tilde{\omega}_j]\, t)\Big\}. \tag{35}$$

The second order correction $\psi_2$ in longitude is computed in a similar way. The corrections on the l.h.s. of (25) come from the first order nutation in obliquity denoted by $\bar{\theta}_1$ (the precession in obliquity is neglected here) and a correction to the first order nutation $\psi_1$

$$\sin\theta\,\dot{\psi} \;\approx\; \sin(\theta_0 + \bar{\theta}_1)\,(\dot{\psi}_1 + \dot{\psi}_2) \approx \approx\; \underline{\sin\theta_0\dot{\psi}_1} + \bar{\theta}_1\cos\theta_0\dot{\psi}_1 + \sin\theta_0\dot{\psi}_2. \tag{36}$$

The underlined term is of first order, while the other two terms describe second order contributions. There is also one term from the expansion of the right hand side of (25) with respect to the periodic correction in $\varphi_2$ (see Eq. (33)). Omitting the detailed derivation, we find in total three contributions. First, the first order solution (29) with all $\omega_{30}[\Delta\omega_i]$ being replaced by $\tilde{\omega}_{30}[\Delta\tilde{\omega}_i]$. Second, what we can call the "precession on nutation" effect in longitude (coming from the $\bar{\theta}_1$ term in (36))

$$\sin\theta_0\,\psi_2^{(1)} = \sum_{i,j} - \cot\theta_0 \bar{a}_3 \frac{b_{1i}}{\Delta\tilde{\omega}_i^{\,2}} \sin(\Delta\phi_i + \Delta\tilde{\omega}_i\, t). \tag{37}$$

Third, the nutation term at the second order in longitude which comes from the remaining second order terms:

$$\sin\theta_0\,\psi_2^{(2)} \;=\; \sum_{i,j} \frac{\cot\theta_0}{2} \frac{a_{3i}}{\Delta\tilde{\omega}_i} \frac{a_{3j}}{\Delta\tilde{\omega}_j} \times \times \sin(\Delta\phi_i + \Delta\phi_j + [\Delta\tilde{\omega}_i + \Delta\tilde{\omega}_j]\, t). \tag{38}$$

Summing up the results from the previous subsections, the complete analytical solution for the Euler angles $\psi, \theta, \varphi$ between the figure axis system and an inertial system for the motion of the Earth according to a (mainly luni-solar) torque of the form (12) has been computed for our level of accuracy. All of the second order terms are proportional to the product of two tidal amplitudes by definition.

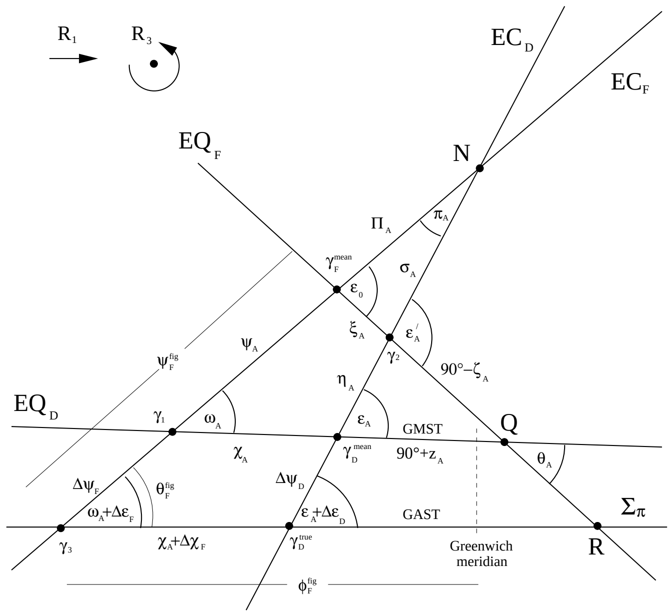


**Fig. 1.** Coordinate systems and variables for precession and nutation

### *2.6. Coordinate systems for precession and nutation*

The definition of the precessional and nutational quantities can be seen in Fig. 1. The subscripts $D$ and $F$ stand for *date* and *fixed* in the following. Important coordinate systems are:

1. the mean equator of date, $\mathrm{EQ_D}$,
2. the mean fixed equator J2000, $\mathrm{EQ_F}$,
3. the mean ecliptic of date, $\mathrm{EC_D}$,
4. the mean fixed ecliptic J2000, $\mathrm{EC_F}$
5. the system related to the figure axis of the Earth, denoted by $\sum_\pi$ here, which is identical to the true equator of date counting the origins from the axis of the smallest moment of inertia $A$ or – by using the axisymmetric approximation for the Earth and a constant rotation which can be included in $\varphi_0$ (see Eq. (19)) – from Greenwich Meridian.

Williams (1994) presents four methods to make the transformation from the mean fixed equator and equinox J2000 (e.g. used in JPL ephemerides DE200) via the mean equinox of date $\gamma_{\mathrm{D}}^{\mathrm{mean}}$ to the true equator and equinox of date. $R_i(\alpha)$ denotes a rotation with angle $\alpha$ around the $i$-axis. For the meaning of the various variables see also Williams (1994) and Fig. 1.

$1^{\rm st}$ method: $(\gamma_{\rm F}^{\rm mean} \rightarrow Q \rightarrow \gamma_{\rm D}^{\rm mean} \rightarrow \gamma_{\rm D}^{\rm true})$

$$\underbrace{R_1(-\varepsilon_A - \Delta\varepsilon_{\rm D})\, R_3(-\Delta\psi_{\rm D})\, R_1(\varepsilon_{\rm A})}_{\mathcal{N}} \times \tag{39}$$

$$\times \underbrace{R_3(-90^\circ - z_{\rm A})\, R_1(\theta_{\rm A})\, R_3(90^\circ - \zeta_{\rm A})}_{\mathcal{P}},$$

where $\mathcal{N}$ and $\mathcal{P}$ are the usual nutation and precession matrix, respectively.

$2^{\rm nd}$ method: $(\gamma_{\rm F}^{\rm mean} \rightarrow \gamma_1 \rightarrow \gamma_{\rm D}^{\rm mean} \rightarrow \gamma_{\rm D}^{\rm true})$

$$R_1(-\varepsilon_{\rm A} - \Delta\varepsilon_{\rm D}) R_3(-\Delta\psi_{\rm D}) R_1(\varepsilon_{\rm A}) \times \tag{40}$$

$$\times R_3(\chi_{\rm A}) R_1(-\omega_{\rm A}) R_3(-\psi_{\rm A}) R_1(\varepsilon_0).$$

$3^{\rm rd}$ method: $(\gamma_{\rm F}^{\rm mean} \rightarrow N \rightarrow \gamma_{\rm D}^{\rm mean} \rightarrow \gamma_{\rm D}^{\rm true})$

$$R_1(-\varepsilon_{\rm A} - \Delta\varepsilon_{\rm D}) R_3(-\Pi_{\rm A} - p_{\rm A} - \Delta\psi_{\rm D}) \times \tag{41}$$

$$\times R_1(\pi_{\rm A})\, R_3(\Pi_{\rm A})\, R_1(\varepsilon_0),$$

where $\Pi_{\rm A} + p_{\rm A} = \sigma_{\rm A} + \eta_{\rm A}$, see Williams (1994).

$4^{\rm th}$ method: $(\gamma_{\rm F}^{\rm mean} \rightarrow \gamma_2 \rightarrow \gamma_{\rm D}^{\rm mean} \rightarrow \gamma_{\rm D}^{\rm true})$

$$R_1(-\varepsilon_{\rm A} - \Delta\varepsilon_{\rm D}) R_3(-\eta_{\rm A} - \Delta\psi_{\rm D}) R_1(\varepsilon'_{\rm A}) R_3(\xi_{\rm A}). \tag{42}$$

And we have another method for later purpose in Sect. 2.8:

$5^{\rm th}$ method: $(\gamma_{\rm F}^{\rm mean} \rightarrow \gamma_3 \rightarrow \gamma_{\rm D}^{\rm true})$

$$R_3(\chi_{\rm A} + \Delta\chi_{\rm F}) R_1(-\varepsilon_{\rm A} - \Delta\varepsilon_{\rm F}) \times \tag{43}$$

$$\times R_3(-\psi_{\rm A} - \Delta\psi_{\rm F}) R_1(\varepsilon_0).$$

These transformations are needed to relate the Euler angles computed in the last subsection to the nutation quantities used in the nutation theories.

### *2.7. The relation between the Euler angles and the nutation and precession variables with respect to the mean fixed ecliptic*

From the previous subsection and with the help of Fig. 1 we can extract the nutation quantities from the Euler angles shown above. Since the Euler angles of the figure axes system directly transform from $\sum_\pi$ to the inertial system represented by the mean fixed ecliptic J2000 (and vice versa) one finds by comparing

$$R_3(\phi) R_1(-\theta) R_3(-\psi) R_1(\varepsilon_0) \tag{44}$$

with (43), or regarding Fig. 1, that the following relations hold

$$\begin{aligned} \psi(T) &\equiv \psi_{\rm F}^{\rm fig} = \psi_{\rm A} + \Delta\psi_{\rm F} \\ \theta(T) &\equiv \theta_{\rm F}^{\rm fig} = \omega_{\rm A} + \Delta\varepsilon_{\rm F} \\ \phi(T) &\equiv \phi_{\rm F}^{\rm fig} = \chi_{\rm A} + \Delta\chi_{\rm F} + {\rm GAST}(T). \end{aligned} \tag{45}$$

$\psi_{\rm A}, \omega_{\rm A}, \chi_{\rm A}$ are respectively the precession in longitude, in obliquity and the planetary precession. $T$ is the time in Julian centuries measured from J2000. Thus there is a very simple relationship between the Euler angles and the nutation angles $\Delta\psi_{\rm F}, \Delta\varepsilon_{\rm F}$ referred to the mean fixed ecliptic J2000. However, traditionally the nutation series are referred to the mean ecliptic of date so the transformation described in the next section is needed.

### *2.8. The relation between the nutation angles referred to the mean fixed ecliptic and those referred to the mean ecliptic of date*

The definition of the nutation angles with respect to the mean ecliptic of date can also be seen in Fig. 1. Thus, $\Delta\psi_{\rm D}$ and $\Delta\varepsilon_{\rm D}$ transform from mean EC$_{\rm D}$ to $\sum_\pi$.

Comparing the transformation (43) to anyone of the four methods (39–42) and solving for $\Delta\psi_{\rm D}$ and $\Delta\varepsilon_{\rm D}$ we obtain (using the numerical values for the other precessional quantities given in Simon et al. 1994)

$$\begin{aligned} \Delta\psi_{\rm D} = \ & \Delta\psi_{\rm F}\,(1 + 0.000\,523\,441\,T + 0.000\,000\,874\,T^2 - \\ & -0.000\,000\,108\,T^3) + \\ & +\Delta\varepsilon_{\rm F}\,(\,+0.000\,128\,678\,T - 0.000\,028\,957\,T^2 - \\ & -0.000\,000\,015\,T^3) \end{aligned} \tag{46}$$

$$\begin{aligned} \Delta\varepsilon_{\rm D} = \ & \Delta\psi_{\rm F}\,(\,-0.000\,020\,360\,T + 0.000\,004\,592\,T^2 + \\ & +0.000\,000\,002\,T^3) + \\ & +\Delta\varepsilon_{\rm F}\,(1 - 0.000\,000\,003\,T^2 - 0.000\,000\,001\,T^3). \end{aligned} \tag{47}$$

Therefore, only the small time-dependent part of the nutation angles is affected by this transformation while the constant part remains unchanged. Whether the higher powers of $T$ produce any significant terms is discussed in Sect 3.5.

### *2.9. The geodesic precession and nutation*

For completeness the relativistic contribution to precession and nutation is also given here. Using the results of Fukushima (1991) or of Damour et al. (1993) and the numerical expressions of Simon et al. (1994) we obtain for the additional terms in longitude (there are no terms in obliquity)

$$\Psi = \frac{3GM_\odot}{2\,c^2 a}\left[(1+e^2) l_{\rm s} + 3\,e \sin l_{\rm s} + \frac{9}{4} e^2 \sin 2l_{\rm s} \ldots\right],$$

where $a$, $e$, and $l_{\rm s}$ are respectively the semi-major axis, eccentricity, and mean anomaly of the Earth's orbit.

$$\begin{aligned} \Psi = \ & 0''\!.019\,062 + 1''\!.919\,336\,T + \\ & +0''\!.000\,153\,1 \sin l_{\rm s} + 0''\!.000\,001\,9 \sin 2l_{\rm s}. \end{aligned} \tag{48}$$

The secular term is the geodesic precession $p_{\rm g}$ and the periodic terms are called the geodesic nutation $\Delta\psi_{\rm g}$. Considering that the kinematically non-rotating reference system is to be adopted these terms should be subtracted from the precession and from the nutation series, respectively.

### *2.10. The other axes – the Oppolzer terms*

The theories of nutation mentioned in Sect. 2.1 compute the nutation of the angular momentum axis and give the transformation to obtain the nutation of the other two axes (figure axis and rotation axis) by adding the so called

Oppolzer terms. The nutation theory presented here which starts from the tidal potential gives the nutation of the figure axis. To get the nutation for the other two axes we have therefore to use some combination of the classical Oppolzer terms

$$\Delta\psi_{\text{fig}\to\text{rot}} = \Delta\psi^{\text{Opp}}_{\text{rot}} - \Delta\psi^{\text{Opp}}_{\text{fig}},$$
$$\Delta\varepsilon_{\text{fig}\to\text{rot}} = \Delta\varepsilon^{\text{Opp}}_{\text{rot}} - \Delta\varepsilon^{\text{Opp}}_{\text{fig}} \quad (49)$$

(see K77 and ZG89). Using some geometrical relations, and vanishing the terms under our threshold, the result derived in Hartmann (1996) is

$$\Delta\psi_{\text{fig}\to\text{rot}} = -\frac{\Delta\tilde{\omega}_i}{\tilde{\omega}_{30}}\frac{\varepsilon_{\text{in}}}{\sin\varepsilon_0}\sin(\Delta\phi_i + \Delta\omega_i t), \quad (50)$$

$$\Delta\varepsilon_{\text{fig}\to\text{rot}} = -\frac{\Delta\tilde{\omega}_i}{\tilde{\omega}_{30}}\psi_{\text{in}}\sin\varepsilon_0\cos(\Delta\phi_i + \Delta\omega_i t) - -\frac{p_A}{\tilde{\omega}_{30}}\sin\varepsilon_0, \quad (51)$$

where $\psi_{\text{in}}$ and $\varepsilon_{\text{in}}$ denote the in-phase terms only.

To get the nutation for the angular momentum axis we have to replace $\tilde{\omega}_{30}$ in Eqs. (50–51) by $C/A\cdot\tilde{\omega}_{30}$. The constant offset in obliquity due to the precession in longitude $p_{\text{A}}$ ($-8\,711.90$ $\mu$as and $-8\,683.38$ $\mu$as for the rotation axis and angular momentum axis, respectively) is well-known, see e.g. Woolard (1953) or Capitaine et al. (1985). For another way to derive the Oppolzer terms, except for the constant offset, see ZG89.

## 3. Results

### *3.1. General remarks*

This section deals with the presentation of the results of our new rigid Earth nutation model called H96NUT computed from the tidal development HW95 using the formulas given in the previous section. The truncation threshold chosen for the nutation amplitudes is 0.45 $\mu$as, which is smaller by one order of magnitude than that from the KS90 nutation model (Kinoshita & Souchay 1990). Thus the condition for keeping a term is:

$$|\sin\varepsilon_0\cdot\Delta\psi| \geq 0.45\ \mu\text{as} \quad \text{or} \quad |\Delta\varepsilon| \geq 0.45\ \mu\text{as}. \quad (52)$$

The factor $\sin\varepsilon_0$ has approximately the value 0.397 777 and is introduced because VLBI measures the product $\sin\varepsilon_0\cdot\Delta\psi$ (see also the discussion in Williams 1995). If only one of the two conditions in (52) is satisfied the other nutation component is also considered. This leads to a total number of 699 nutation terms, both in longitude and obliquity. According to Williams (1995) or any textbook on statistics the summation of the whole nutation series introduces some error due to the limited number of digits for the nutation amplitudes. The total error is at least

$$\sigma_{\text{total}} = \sqrt{\frac{N}{2}}\,\sigma_i. \quad (53)$$

where $\sigma_i$ is the individual error and $N$ the number of terms. Since $N = 699$ for H96NUT this error is larger by a factor of 19 than the individual error. To keep that error under the truncation threshold of 0.45 $\mu$as all nutation amplitudes are given to a resolution of 0.01 $\mu$as. The accuracy of the individual nutation terms is discussed in more detail below.

**Table 1.** Precession and obliquity rates at J2000 from direct planetary torques on the Earth's bulge. (in $\mu$as yr$^{-1}$), $H_{\text{dyn}} = 3.273\,792\,489\ 10^{-3}$. The subscript $H$ denotes H96NUT, $W$ Williams (1995), respectively

| body | $\Delta\dot{\psi}_{\text{H}}$ | $\Delta\dot{\psi}_{\text{W}}$ | $\Delta\dot{\varepsilon}_{\text{H}}$ | $\Delta\dot{\varepsilon}_{\text{W}}$ |
|---|---|---|---|---|
| Mercury | 3.700 | 3.697 | −0.089 | −0.088 |
| Venus | 181.586 | 181.565 | −16.799 | −16.813 |
| Mars | 6.001 | 5.998 | 0.356 | 0.356 |
| Jupiter | 117.050 | 117.068 | 2.810 | 2.804 |
| Saturn | 5.211 | 5.188 | 0.220 | 0.219 |
| Uranus | 0.098 | 0.100 | 0.000 | 0.001 |
| Neptune | 0.028 | 0.029 | 0.000 | 0.001 |
| Total | 313.674 | 313.645 | −13.502 | −13.520 |

### *3.2. Planetary and luni-solar precession*

As indicated in the last section it is also possible to deduce values for the precession from the amplitudes of the $K_1$-tide. Using the tidal model HW95 (and the separately derived values for the $K_1$-tide due to Uranus and Neptune) results for the planetary effect on the general precession in longitude and on the obliquity rate are given in Table 1 showing excellent agreement with the values given by Williams (1995).

The results in longitude and obliquity for the luni-solar precession are shown in Table 2. The agreement in longitude is within the current accuracy of the observations (about five digits). However, for the precession in obliquity the numbers coming from the tidal potential are much less accurate. This is due to the fact that they arise due to a smaller tide very close to the $K_1$-tide and not to the $K_1$-tide itself. Note that the best value for the precession in obliquity is still under discussion.

### *3.3. The nutation terms at the first order*

Next, some of the results for the nutation are presented. Most of the numbers refer to the nutation of the figure axis. First there is an overview in Table 3 on the terms due to the different effects taken into account.

#### 3.3.1. Direct planetary nutation terms

The direct planetary nutation terms have been first computed by Vondrák (1982) and then they have been included in most of the nutation theories computed in the nineties. Hartmann & Soffel (1994) computed the nutation

**Table 2.** Luni-solar precession in longitude and obliquity (in $\mu$as yr$^{-1}$, $H_{\rm dyn} = 3.273\,792\,489\ 10^{-3}$)

| body (+degree $\ell$) | tidal frequency [° / h] | period [years] | in longitude $\times T$ | $\times T^2$ | in obliquity $\times T$ | $\times T^2$ |
|---|---|---|---|---|---|---|
| Mo2: | 15.041 068 64 | Infinity | 34 449 009.598 | −6 675.879 | 0.023 | 0 |
| Mo2: | 15.041 072 56 | 10 465 | 5 915.430 | −646.162 | 1 141.284 | −37.439 |
| Mo3: | 15.041 066 68 | 20 930 | −0.032 | 0 | −0.003 | 0 |
| Mo3: | 15.041 070 60 | 20 930 | 0.089 | 0 | −0.008 | 0 |
| Mo4: | 15.041 068 64 | Infinity | 25.152 | 0 | 0 | 0 |
| Mo: | | | 34 454 950.238 | −7 322.041 | 1 141.307 | −37.439 |
| Su2: | 15.041 068 64 | Infinity | 15 945 490.664 | −3 060.036 | 0.011 | 0 |
| Su2: | 15.041 064 72 | 10 465 | 3 370.901 | −347.260 | 650.359 | −17.292 |
| Su3: | 15.041 066 68 | 20 930 | −0.015 | 0 | −0.001 | 0 |
| Su3: | 15.041 070 60 | 20 930 | 0.042 | 0 | −0.004 | 0 |
| Su: | | | 15 948 861.592 | −3 407.299 | 650.365 | −17.292 |
| $p_{\rm g}$ | | | −19 193.360 | | | |
| Hartmann | (1996) | | 50 384 618.470 | −10 729.340 | 1 791.672 | −54.731 |
| Mo: | | | 34 454 718.000 | | | |
| Su: | | | 15 948 722.000 | | | |
| Williams | (1994) | | 50 384 246.000 | −10 789.770 | −244.000 | 512.680 |
| Simon et al. | (1994) | | 50 385 064.600 | −10 723.810 | 0.000 | 512.940 |

**Table 3.** Number of nutation ($\mathcal{N}$) and precession ($\mathcal{P}$) terms in H96NUT, truncation level at 0.45 $\mu$as

| body | $\ell$ | $\mathcal{P}$ | $\mathcal{N}$ | remarks |
|---|---|---|---|---|
| Mercury | 2 | 1 | 0 | direct nutation |
| Venus | 2 | 1 | 41 | |
| Mars | 2 | 1 | 14 | |
| Jupiter | 2 | 1 | 10 | |
| Saturn | 2 | 1 | 3 | |
| Uranus | 2 | 1 | 0 | |
| Neptune | 2 | 1 | 0 | |
| Moon | 4 | 1 | 1 | due to $J_4^{\oplus}$ |
| Moon | 3 | 2 | 11 | due to $J_3^{\oplus}$ |
| Sun | 3 | 2 | 0 | |
| Moon | 2 | 2 | 7 | triax. terms |
| | | | 30 | indirect nutation |
| | | | 429 | direct nutation |
| Sun | 2 | 2 | 2 | triax. terms |
| | | | 88 | indirect nutation |
| | | | 28 | direct nutation |
| Moon + Sun | 2 | 0 | 35 | second order terms |
| H96NUT | | 16 | 699 | |

due to the direct planetary effect using a simplified version of the theory outlined in the last section. The planetary tidal potential used at that time was based on the numerical ephemerides DE102 but differences to DE200 are very small. The threshold was 2.5 $\mu$as and – as in Williams (1995) where it is 0.5 $\mu$as – the same for $\Delta\psi$ and $\Delta\varepsilon$ thus ignoring the factor $\sin\varepsilon_0$. For H96NUT the HW95 tidal potential based on DE200 and a threshold of 0.45 $\mu$as was chosen. A term by term comparison gives for the maximum difference, the sum of the absolute differences and

**Table 4.** Term by term comparison: direct planetary nutation H96NUT – Williams (1995), values in $\mu$as

| | $\Delta\psi_{\rm sin}$ | $\Delta\psi_{\rm cos}$ | $\Delta\varepsilon_{\rm sin}$ | $\Delta\varepsilon_{\rm cos}$ |
|---|---|---|---|---|
| Max. | 2.31 | 3.77 | 1.26 | 1.50 |
| $\sum\lvert\ \rvert$ | 13.21 | 16.14 | 6.12 | 5.87 |
| rms | 0.35 | 0.53 | 0.18 | 0.20 |

**Table 5.** Comparison H96NUT – Williams (1995) in time domain: direct planetary nutation, values in $\mu$as

| | Max | Min | Mean | rms |
|---|---|---|---|---|
| $\Delta\psi$ | 8.26 | −12.14 | −1.47 | 3.53 |
| $\Delta\varepsilon$ | 4.75 | −1.34 | 1.57 | 1.82 |

the root mean squares (rms) error the values indicated in Table 4. Now, computing a time series over 90 years starting from J2000 and evaluating the differences in time domain leads to the results in Table 5. It is important to note that the rms values are about one order of magnitude larger than those of the term by term comparison. Furthermore, the maximum and minimum differences (see Table 5) nearly reach the sum of the absolute differences of the individual terms (see Table 4). As already mentioned in Hartmann & Soffel (1994) the direct planetary nutation can be computed from a tidal potential with high accuracy (also due to the smallness of these terms).

### 3.3.2. Nutation term due to $J_3$ and $J_4$

The nutation terms due to the higher order parts of the tidal potential have already been given in Hartmann et al.

**Table 6.** The triaxiality terms, values in $\mu$as, period in days. The cut-off level considered for the table corresponds to the condition (52). The subscripts $H$ and $RD$ stand for H96NUT and Roosbeek & Dehant (1997), respectively

| $l_m$ | $l_s$ | $F$ | $D$ | $\Omega$ | period | $\Delta\psi_H$ | $\Delta\psi_{RD}$ | $\Delta\varepsilon_H$ | $\Delta\varepsilon_{RD}$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | −4 | 0 | −4 | 0.538 | | −1.43 | | 0.57 |
| −1 | 0 | −2 | 0 | −2 | 0.527 | −5.94 | −2.46 | 2.36 | 0.98 |
| 0 | 0 | −2 | 0 | −1 | 0.518 | −5.52 | −5.61 | 2.20 | 2.23 |
| 0 | 0 | −2 | 0 | −2 | 0.518 | −29.25 | −28.31 | 11.63 | 11.26 |
| 1 | 0 | −2 | 0 | −2 | 0.508 | | −2.19 | | 0.87 |
| −1 | 0 | 0 | 0 | 0 | 0.508 | 2.17 | −0.79 | −0.86 | 0.31 |
| 0 | 0 | −2 | 2 | −2 | 0.500 | −12.23 | −12.14 | 4.87 | 4.83 |
| 0 | 0 | 0 | 0 | 0 | 0.499 | 36.72 | 38.50 | −15.60 | −15.32 |
| 0 | 0 | 0 | 0 | −1 | 0.499 | 4.98 | 4.95 | −1.98 | −1.97 |
| 1 | 0 | 0 | 0 | 0 | 0.490 | 1.95 | 4.77 | −0.77 | −1.90 |

(1996). As an extension the largest solar term due to $J_3^{\oplus}$ has the argument $F - D + \Omega$, a period of 365.229 days and amplitudes of $-0.26$ $\mu$as and $-0.22$ $\mu$as for $\Delta\psi$ and $\Delta\varepsilon$, respectively. Hence, this term is under our threshold of 0.45 $\mu$as.

#### 3.3.3. Nutation terms due to the triaxiality of the Earth

The nutation terms due to the triaxiality of the Earth come from the difference between the equatorial moments of inertia $A$ and $B$ (or equivalently from the geopotential coefficients $C_{22}$ and $S_{22}$). Since that difference is small and the denominator in the nutation formulas (28–30) is not the small difference $\Delta\tilde{\omega}_i$ but the large sum $\Sigma\tilde{\omega}_i$, only few terms play a role. Some authors do not consider them due to their short periods of about half a day. Nevertheless, they have been calculated by Kinoshita & Souchay (1990), with better accuracy by Souchay & Kinoshita (1997a,b) and by Roosbeek & Dehant (1997). The terms are given in Table 6, where $2\tau$ defined in Eq. (9) has to be added to the argument, and a comparison with RDAN97 shows that most of the terms are in good agreement with respect to the thresholds.

#### 3.3.4. Luni-solar nutation terms at first order due to $J_2$

The direct luni-solar nutation terms, i.e. those which can be represented solely by the Delaunay arguments $l_m, l_s, F, D$ and $\Omega$, and the indirect planetary nutation terms which contain besides at least one mean longitude of a planet are calculated the same way from the tidal potential HW95. From Table 3 we can see that these terms make up more than 80% of all terms in H96NUT. The term by term comparison of H96NUT and RDAN97 is presented in Table 11.

For the transformation between the tidal argument and the nutational argument see ZG89, p. 1107. It must be mentioned that there is some ambiguity concerning the sign of nutation amplitudes and the nutation argument. Here, all signs were chosen in such a unique way that the resulting nutation period is positive. It is hoped that other authors will follow that convention to simplify comparisons between different nutation models.

**Table 7.** $T^2$-contributions to the nutation amplitudes

| Argument | | | | | Period | $T^2 \cdot \Delta\psi_{\sin}$ | $T^2 \cdot \Delta\varepsilon_{\cos}$ |
|---|---|---|---|---|---|---|---|
| $l_m$ | $l_s$ | $F$ | $D$ | $\Omega$ | [days] | [$\mu$as cy$^{-2}$] | [$\mu$as cy$^{-2}$] |
| 0 | 0 | 0 | 0 | −1 | 6798.383 | −248.05 | 79.19 |
| 0 | 0 | 0 | 0 | −2 | 3399.192 | 2.48 | −0.96 |
| 0 | 0 | 2 | −2 | 2 | 182.621 | −16.84 | −5.87 |
| 0 | 0 | 2 | 0 | 2 | 13.661 | −2.89 | −1.02 |

### *3.4. The nutation terms at second order*

The additional contributions not considered at first order are: the effect of the precession on the nutation as given in (37), the geodesic nutation from (48) and the second order terms according to (38) and (35). It must be stressed that these terms cannot be compared directly to second order terms of other nutation series since they differ by definition. To demonstrate this important fact let us consider the so-called $J_2$-tilt effect. The $J_2$ coefficient of the Earth slightly tilts the orbit of the Moon with respect to the ecliptic. This is implicitly included in the tidal potential HW95 and therefore also in the nutation model H96NUT (due to the inclusion of the corresponding force and the numerical integration of the equations of motion of all bodies in the solar system in the numerical ephemerides DE200). Other theories which use analytical ephemerides without this force have to include the $J_2$-tilt effect separately. However, the total nutation amplitudes should be directly comparable.

### *3.5. The influence of $T^2$ terms*

Another point that deserves attention is the influence of $T^2$ terms. There are basically two points where they appear. First, there are the nutation amplitudes themselves. Since the tidal amplitudes are only linear with respect to time $T$ the same applies to the Euler angles. However, the transformation from the fixed ecliptic to the ecliptic of date introduces higher powers of $T$ which might not be neglected if high accuracy is required. For the model H96NUT $T^2$-contributions to the nutation amplitudes are listed in Table 7. For small time intervals around J2000 they may be omitted.

The second place where $T^2$-contributions occur is in the nutation arguments. On the assumption that the arguments take the form $\arg(T) = a_0 + a_1 T + a_2 T^2$ in all nutation theories an integration with respect to time is required to solve the rotational equations of motion.

**Table 8.** Corrections due to $T^2$-terms in the nutation arguments

| Argument | | | | | Period | $T \cdot \Delta\psi_{\sin}$ | $\Delta\psi_{\cos}$ |
|---|---|---|---|---|---|---|---|
| $l_m$ | $l_s$ | $F$ | $D$ | $\Omega$ | [days] | [$\mu$as cy$^{-1}$] | [$\mu$as] |
| 0 | 0 | 0 | 0 | −1 | 6798.383 | −37.09 | −1.10 |
| 0 | 0 | 0 | 0 | −2 | 3399.192 | 0.45 | 0.01 |

**Table 9.** Term by term comparison: H96NUT – RDAN97, values in $\mu$as. Max. are the absolute values

| | $\Delta\psi_{\sin}$ | $\Delta\psi_{\cos}$ | $t\Delta\psi_{\sin}$ | $t\Delta\psi_{\cos}$ |
|---|---|---|---|---|
| Max. | 5 355.52 | 159.53 | 736.91 | 56.79 |
| $\sum\|\ \|$ | 8 686.11 | 659.90 | 1 494.97 | 137.66 |
| rms | 237.33 | 9.66 | 35.82 | 2.46 |
| | $\Delta\varepsilon_{\sin}$ | $\Delta\varepsilon_{\cos}$ | $t\Delta\varepsilon_{\sin}$ | $t\Delta\varepsilon_{\cos}$ |
| Max. | 140.76 | 1 788.89 | 7.27 | 558.78 |
| $\sum\|\ \|$ | 265.46 | 3 893.92 | 29.11 | 847.64 |
| rms | 6.00 | 100.22 | 0.43 | 24.72 |

**Table 10.** Percentage distribution of residuals ($\Delta$) in intervals from the term by term comparison. The boundaries of intervals are in $\mu$as. There is total number 580 of the compared terms

| | $\Delta \leq 5$ | $2 \leq \Delta < 5$ | $1 \leq \Delta < 2$ | $\Delta < 1$ |
|---|---|---|---|---|
| $\Delta\psi_{\sin}$ | 4.8 | 4.5 | 4.3 | 86.4 |
| $\Delta\psi_{\cos}$ | 3.1 | 2.3 | 2.9 | 91.7 |
| $t\Delta\psi_{\sin}$ | 1.9 | 1.5 | 1.4 | 95.2 |
| $t\Delta\psi_{\cos}$ | 0.7 | 0.5 | 2.1 | 96.7 |
| $\Delta\varepsilon_{\sin}$ | 1.2 | 1.7 | 1.0 | 96.1 |
| $\Delta\varepsilon_{\cos}$ | 3.8 | 1.4 | 3.1 | 91.7 |
| $t\Delta\varepsilon_{\sin}$ | 0.3 | 0.2 | 0.2 | 99.3 |
| $t\Delta\varepsilon_{\cos}$ | 1.0 | 0.9 | 0.9 | 97.2 |

Then, approximating the integral $\int \cos(\arg(T))\,\mathrm{d}T$ by $1/a_1 \sin(\arg(T))$ results in an error

$$\int \cos(\arg(T))\,\mathrm{d}T - \frac{1}{a_1}\sin(\arg(T)) \approx -\frac{2\,a_2}{a_1^2}\left[T\,\sin(\arg(T)) + \frac{1}{a_1}\cos(\arg(T))\right]. \quad (54)$$

For the model H96NUT only the 18.6 and 9.3 yr term (see Table 8) contribute significantly, (after a long time) thus proving the heuristic linear approximation is quite appropriate.

Using only the linearized nutation argument results in an error increasing quadratically with time. For the main nutation term in longitude (amplitude 17″) this can reach values up to 622 $\mu$as after one century. Thus at least the $T^2$ coefficients – or as in H96NUT up to the $T^4$ coefficient – for the astronomical arguments must be taken into account.

## 4. Error estimation

This section tries to give some error estimates for the individual nutation terms while the next section is devoted to the error on the nutation given by the whole nutation series. Clearly, one has to consider the largest terms only. The first point to be mentioned is that the tidal potential HW95 has been computed with a fixed amplitude threshold. The approximate amplitude of a (first order) nutation term is given by

$$A_{\mathrm{nut}} = |\sin\varepsilon_0\,\Delta\psi| = |\Delta\varepsilon| = \left|\frac{a_3}{\Delta\tilde{\omega}_i}\right| \approx \frac{S_i^{21}}{\Delta\tilde{\omega}_i}\frac{H_{\mathrm{dyn}}}{\omega_i} + \ldots \quad (55)$$

Due to the resonance denominator $\Delta\tilde{\omega}_i = \omega_i - \tilde{w}_{30}$ nutation terms might arise even when the tidal amplitude $S_i^{21}$ (see Eq. (8)) is small (or even below the chosen threshold for the tidal potential) provided the tidal frequency $\omega_i$ is close enough to the $K_1$-tide for which $\Delta\tilde{\omega}_i \equiv 0$. Using the numbers for the smallest tidal amplitudes from the HW95 model one finds that nutation terms of the order 0.45 $\mu$as might be missed for periods longer than 350 days for the Moon and 3.6 years for the Sun. Obviously, that problem could be avoided if the threshold for the tidal potential would have been chosen to be frequency dependent as was suggested by Dehant et al. (1997). Moreover, the transfer function between the rigid Earth nutation and a more realistic Earth nutation has some other resonances (like the free core nutation (FCN) at about 430 days) in the diurnal tidal band which might also amplify small tidal rigid Earth nutation amplitudes to give a real nutation term above the threshold. Originally we hoped to achieve such a small threshold for the tidal amplitudes that any frequency dependence is unnecessary. Unfortunately, due to the numerical computation method for the tidal potential HW95 that could not be achieved. The frequency resolution was – in spite of all efforts – not high enough to resolve tides which are very close to the $K_1$-tide or to other large tidal terms. It should be noted that the threshold for the tidal amplitudes is smaller by a factor of about 3–5 in comparison with accurate recent tidal model RATGP95 of Roosbeek (1996). Since the factor $\tilde{w}_{30}/\Delta\tilde{\omega}_i$ can reach values up to $10^6$ this is – besides the HW95 tidal model – the only one which is precise enough for the computation of nutation series to highest accuracy using the method outlined in this paper.

Next, the influence of possible errors in the HW95 model on the corresponding nutation series H96NUT is investigated. Assuming the numerical analysis procedure missed the correct tidal frequency $\omega_i$ in (55) by $\delta\omega$ this yields an error for the nutation amplitude of about

$$\delta A_{\mathrm{nut}} \approx A_{\mathrm{nut}}\frac{-\delta\omega}{\Delta\omega_i} = A_{\mathrm{nut}}\frac{-\delta\omega}{2\pi}P_i, \quad (56)$$

where $P_i$ is the period of the $i$-th term in Julian centuries. Therefore, this error grows with the period of the nutation term thus confirming the statement of ZG89 and Souchay (1993) that long-periodic nutation terms have larger errors than the short-periodic ones. However, this affects only the indirect planetary tides, and corresponding nutation terms. As for the other tides, the correct frequency could be found. Estimating the probable frequency uncertainty not to be larger than 1.35 rad $\mathrm{cy}^{-1}$, it follows that an indirect planetary nutation term with a period of 18.6 and 250 years has a relative uncertainty of 4% and 50%, respectively.

In addition, there are some tides with very small tidal amplitudes which lead to nutation terms only because the frequency is close enough to the $K_1$-tide. Since those tidal amplitudes are of the order of the truncation threshold for the tidal potential, it is doubtful whether these nutation terms are real or appear only because of numerical reason. For the tidal potential HW95 and the nutation series H96NUT this applies again mainly to the long-periodic nutation terms.

There is also another point to be mentioned. In all tidal and nutation models there are terms which differ in argument only by $2p_{\mathrm{s}}$ where $p_{\mathrm{s}}$ is the perigee of the Sun with a period of about 20 000 years. By comparison with other tidal models it turned out that HW95 shows some differences concerning the tidal amplitudes of the terms whose argument differs by $2p_{\mathrm{s}}$, especially when one of the tidal amplitudes is rather large. It seems that the numerical procedure used during the computations, and the final least squares fit of the tidal amplitudes is somewhat critical to these terms. In particular, the rather small tide at $15\overset{\circ}{.}04327897$ in HW95 produces a nutation term in H96NUT with a period of 6 786 days and an amplitude of about 5 200 $\mu$as in longitude which is too large in comparison with other nutation models by nearly 5 mas.

Finally, let us consider the accuracy of the constants involved in the computation of the nutation. Some discussion about that topic can be found in Souchay & Kinoshita (1996, 1997a). The scaling factor for the nutation, namely the dynamical ellipticity $H_{\mathrm{dyn}}$, is currently deduced from the precession constant $p_{\mathrm{A}}$. Based on the theory of KS90, chapter eight, and the precession constant of Simon et al. (1994), $p_{\mathrm{A}} = 5\,028\overset{\prime\prime}{.}82$, the value used for H96NUT is $H_{\mathrm{dyn}} = 3.273\,792\,489\;10^{-3}$. Comparing this with the value found by Williams (1994) $p_{\mathrm{A}} = 5\,028\overset{\prime\prime}{.}77$ and used also in RDAN97 and SMART97, one finds that the relative accuracy is not better than $10^{-5}$ which therefore produces an error in the largest nutation term of about 173 $\mu$as. The next largest nutation term is smaller by almost one order of magnitude. Therefore, one can doubt whether it presently makes sense to compute nutation amplitudes much smaller than 1 $\mu$as.

According to the previous discussions, the conclusion is that short-periodic nutation terms with periods shorter than say 18.6 years except for the largest ones can be computed precisely from the tidal potential HW95 while the long-periodic ones have larger uncertainties.

## 5. Comparison

### *5.1. General remarks*

This section is dedicated to the global comparison of H96NUT with existing nutation series. First of all one has to choose the most suitable model for a comparison. Obviously, the older nutation models (like K77, ZG89 or KS90) have – although they are well established – a threshold which is not comparable to that of H96NUT. The recent theories of the nutation for the rigid Earth model SK96.2, RDAN97, SMART97 as already mentioned in Sect. 1 have a threshold smaller than our model and therefore we have only to decide which of them is the most convenient one for comparison. Unfortunately, there is no suitable benchmark nutation series, that is the total nutation angles computed over some time interval at discrete times, in contrast to the situation with tidal models where at least two reference series exist. The benchmark RDNN97 mentioned in Roosbeek & Dehant (1997) is still under testing and not yet prepared for public use. What makes the comparison more complicated is that in the particular nutation models different variables in the arguments are used. We use in H96NUT the mean longitude of planets referred to the mean dynamical ecliptic and equinox of date. Roosbeek & Dehant (1997) use the mean longitude referred to the mean dynamical ecliptic and equinox J2000. In both theories the values are taken from Simon et al. (1994) and differ by the value of the general precession in longitude $p_{\mathrm{A}}$. Williams (1995), Kinoshita & Souchay (1990), Souchay & Kinoshita (1996, 1997a,b), use the mean longitude referred to the mean dynamical ecliptic and equinox J2000 and moreover, the general precession $p_{\mathrm{A}}$ as an additional fundamental argument. In Bretagnon et al. (1998) the mean longitudes of planets are reckoned from the equinox of date. Their values differ from those of Simon et al. (1994) due to modifications of the tidal model in the lunar theory and the new inertial ecliptic and dynamical equinox defined by DE403/LE403 (Standish et al. 1995). It must be mentioned that also the precession theories used in the computations sometimes differ: Lieske et al. (1977), Williams (1994) or Simon et al. (1994) are commonly in use. Therefore, the scaling factor $H_{\mathrm{dyn}}$ also takes different values but in recent works the agreement is better than the relative uncertainty (about $10^{-5}$) due to that of the precession constant $p_{\mathrm{A}}$. It is also doubtful whether a simple rescaling of the nutation amplitudes makes much sense. To close these general remarks it should be noted that little is said about the consistency of the various nutation models, i.e. whether all theories and numerical constants (e.g. $H_{\mathrm{dyn}}$) used in the computations are compatible with each other or not. Except for the older expression of Aoki et al. (1982) used in the computation of the tidal potential HW95 and the relationship between $H_{\mathrm{dyn}}$ and $p_{\mathrm{A}}$ taken from KS90, everything else is compatible here.

**Table 11.** Differences larger than 5 $\mu$as in longitude $\Delta\psi$ and obliquity $\Delta\varepsilon$ (in $\mu$as) between H96NUT and RDAN97 ($t$ in J cy)

| $l_V$ | $l_{Ma}$ | $l_J$ | $l_{Sa}$ | $l_m$ | $l_s$ | $F$ | $D$ | $\Omega$ | Period | $\Delta\psi_{\sin}$ | $\Delta\psi_{\cos}$ | $t\Delta\psi_{\sin}$ | $t\Delta\psi_{\cos}$ | $\Delta\varepsilon_{\sin}$ | $\Delta\varepsilon_{\cos}$ | $t\Delta\varepsilon_{\sin}$ | $t\Delta\varepsilon_{\cos}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 0 | 0 | 0 | 0 | 0 | −13 | 13 | −13 | 91505.111 | 502.97 | 142.50 | 10.61 | 56.79 | −140.76 | −46.20 | −7.27 | 12.92 |
| 0 | 0 | 0 | 0 | −1 | 0 | 1 | 0 | 3 | 65502.278 | 0.00 | 6.02 | 0.00 | 0.00 | | | | |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 10746.940 | −25.11 | 4.44 | 0.20 | 0.42 | −5.83 | −8.46 | −0.10 | 0.11 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 6798.383 | 1921.63 | −159.53 | 427.96 | −0.29 | −21.49 | 1616.36 | 0.00 | −558.78 |
| 0 | 0 | 0 | 0 | 0 | −2 | 2 | −2 | 1 | 6786.317 | 5355.52 | −21.87 | −736.91 | −0.01 | 6.03 | 1788.89 | 0.00 | −204.11 |
| 8 | 0 | 0 | 0 | 0 | 0 | −13 | 13 | −14 | 6328.227 | −16.52 | −39.02 | −2.31 | 1.55 | 14.53 | −5.90 | 0.00 | −1.23 |
| 0 | 0 | 0 | 0 | −1 | 1 | 0 | 1 | 1 | 6164.101 | 28.86 | 0.00 | 0.08 | 0.00 | 0.00 | 5.29 | 0.00 | 0.00 |
| 0 | 2 | 0 | 0 | 0 | 0 | −1 | 1 | −1 | 5760.478 | 17.04 | −37.09 | −0.45 | 0.89 | 1.86 | 14.07 | 0.00 | 0.00 |
| 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 5373.470 | −4.63 | −9.07 | 0.52 | 0.46 | −6.56 | 3.04 | 0.25 | −0.28 |
| 5 | 0 | 0 | 0 | 0 | 0 | −8 | 8 | −7 | 4951.106 | 6.30 | −5.39 | −6.62 | 0.39 | | | | |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 4330.596 | 10.90 | 57.33 | 0.00 | −0.35 | | | | |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −2 | 3399.192 | −52.37 | −7.75 | −13.85 | 0.00 | 4.03 | −72.44 | 0.00 | 3.16 |
| 0 | 0 | 0 | 0 | 0 | −2 | 2 | −2 | 0 | 3396.173 | −75.92 | 0.00 | −0.04 | 0.00 | 0.00 | −26.63 | 0.00 | 0.00 |
| −3 | 0 | 0 | 0 | 0 | 0 | 5 | −5 | 5 | 2957.350 | 14.19 | −2.23 | −0.17 | 11.65 | −2.07 | −1.75 | 6.29 | 0.42 |
| 0 | 4 | 0 | 0 | 0 | 0 | −2 | 2 | −2 | 2880.239 | −7.01 | 5.57 | 0.65 | 0.01 | | | | |
| 5 | 0 | 0 | 0 | 0 | 0 | −8 | 8 | −8 | 2864.764 | −9.94 | −27.62 | −0.10 | −0.99 | 13.10 | −6.75 | −0.54 | 0.24 |
| 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 2165.298 | −9.17 | 7.78 | 0.97 | 5.53 | | | | |
| 0 | 0 | 0 | 0 | −2 | 0 | 2 | 0 | 2 | 1615.748 | 14.41 | 0.01 | 1.29 | 0.00 | | | | |
| −1 | 0 | 0 | 0 | 0 | 0 | 2 | −2 | 2 | 975.277 | −0.95 | −14.11 | −0.18 | 0.00 | | | | |
| −4 | 0 | 0 | 0 | 0 | 0 | 7 | −7 | 7 | 733.412 | 0.42 | −8.76 | −0.24 | 0.15 | | | | |
| 4 | 0 | 0 | 0 | 0 | 0 | −6 | 6 | −6 | 727.580 | −1.07 | 0.22 | 1.04 | 5.07 | | | | |
| 6 | 0 | 0 | 0 | 0 | 0 | −9 | 9 | −9 | 485.053 | −0.08 | 5.26 | −0.19 | −0.14 | | | | |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 365.260 | 11.38 | 0.56 | -8.00 | 0.00 | 0.37 | 10.69 | 0.00 | −3.56 |
| 0 | 0 | 0 | 0 | 0 | −1 | 2 | −2 | 2 | 365.225 | 16.31 | 0.39 | −6.99 | 0.00 | 0.08 | 11.48 | 0.00 | −2.06 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | −2 | 3 | 187.662 | | | | | 0.00 | −19.90 | 0.00 | 0.00 |
| 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 182.630 | −106.87 | 0.03 | 50.58 | 0.00 | 0.01 | 32.26 | 0.00 | −18.06 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | −2 | 2 | 182.621 | −95.48 | −1.63 | 118.82 | 0.00 | −1.04 | 34.49 | 0.00 | 7.72 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | −2 | 1 | 177.844 | −11.57 | 0.35 | −0.47 | 0.00 | 0.00 | 14.64 | 0.00 | −0.08 |
| 0 | 0 | 0 | 0 | 1 | −1 | 0 | 0 | 1 | 29.934 | 8.89 | 0.00 | 0.10 | 0.00 | | | | |
| 0 | 0 | 0 | 0 | 1 | −2 | 2 | −2 | 2 | 27.554 | | | | | 0.00 | 5.20 | 0.00 | −0.91 |
| −8 | 0 | 0 | 0 | 0 | 0 | 15 | −13 | 15 | 13.663 | −12.90 | −7.79 | −0.01 | 0.00 | −3.17 | 5.08 | 0.00 | 0.00 |
| 0 | 0 | 0 | 0 | 0 | 2 | 0 | 2 | 0 | 13.661 | −69.99 | 0.00 | 14.50 | 0.00 | 0.00 | 27.20 | 0.00 | −5.57 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 2 | 13.661 | −62.61 | 2.77 | 26.81 | 0.00 | 1.47 | 24.19 | 0.00 | −3.90 |
| 8 | 0 | 0 | 0 | 0 | 0 | −11 | 13 | −11 | 13.659 | 13.21 | −7.84 | 0.01 | 0.00 | −3.19 | −5.20 | 0.00 | 0.00 |
| 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 13.633 | −8.73 | −0.61 | 1.53 | 0.00 | 0.00 | 7.87 | 0.00 | 0.58 |
| 0 | 0 | 0 | 0 | 1 | 0 | 2 | 0 | 2 | 9.133 | −9.05 | 0.34 | 2.94 | 0.00 | | | | |

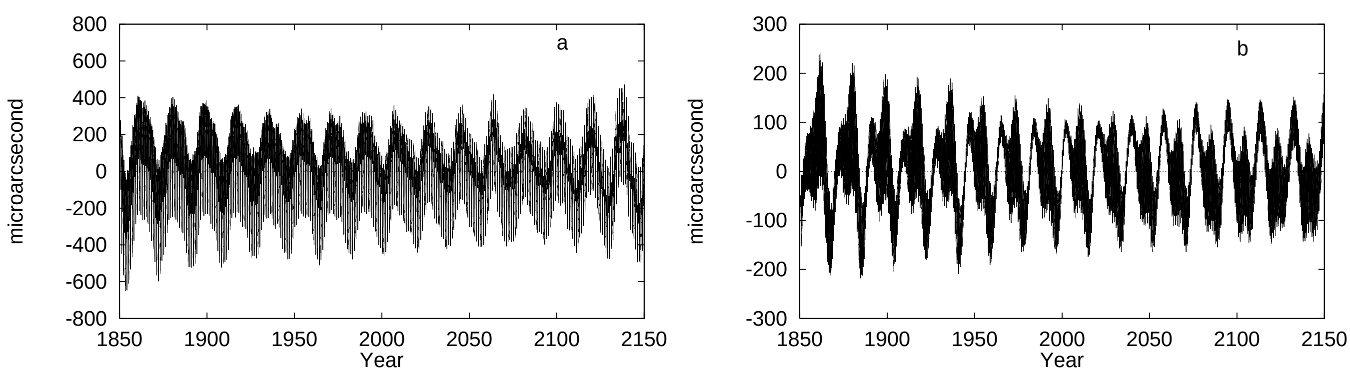


**Fig. 2.** Comparison in time domain H96NUT – SMART97 for short-periodic terms only $\Delta\psi$ **a)** $\Delta\varepsilon$ **b)**: values in $\mu$as

**Table 12.** Overall and short period only comparison in time domain: H96NUT – SMART97, values in $\mu$as

| | Overall | | Short period | |
|---|---|---|---|---|
| | $\Delta\psi$ | $\Delta\varepsilon$ | $\Delta\psi$ | $\Delta\varepsilon$ |
| Max. | 6 317 | 1 459 | 488 | 243 |
| Min. | −5 899 | −1 369 | −660 | −220 |
| rms | 2 815 | 624 | 186 | 74 |

### *5.2. Comparison in detail*

For the term by term comparison we have chosen the model RDAN97 of Roosbeek & Dehant (1997) which appeared recently. This model employed a threshold of 0.1 $\mu$as and contains 1553 terms with similar arguments to that used in H96NUT. While the expressions of Delaunay's arguments $D$, $F$, $l$, $l_s$ and $\Omega$ are the same in both theories, different expressions for the mean longitudes of the planets were used. The mean longitude referred to the mean dynamical ecliptic and equinox of date is used in H96NUT in comparison with the mean longitude referred to the mean dynamical ecliptic and equinox J2000 used in RDAN97. Both systems were taken from Simon et al. (1994) and they differ by the value of the general precession in longitude $p_A$. This difference appears in the direct and indirect planetary terms only and is vanishing for J2000 when the longitudes are the same in both systems. The compensation of the difference in the arguments should appear in the secular term of amplitude in the other phase. For example, for the largest of concerning terms in $\Delta\psi_{\sin}$ having in argument $8l_V$ with an amplitude close to 300 $\mu$as and period 91505.1 days the difference between the series should appear in $t\Delta\psi_{\cos}$ with amplitude about 50 $\mu$as (for $t$ not too large) as we can see in the first term of Table 11. In the comparison we have found 27 terms of H96NUT which are not involved in RDAN97. 14 of them are indirect planetary terms coming from the luni-solar potential, one is a direct planetary term of Venus and remaining 12 ones are luni-solar terms coming from the Moon.

The nutation model SMART97 (Bretagnon et al. 1998) is not convenient for the term by term comparison with H96NUT in spite of having the threshold more than one order smaller because of the different fundamental nutation arguments. To take advantage of the quality of this model we have utilized it for the comparison in the time domain.

### *5.3. Results of the comparison*

First, a term by term comparison for the nutation of the figure axis between H96NUT and RDAN97 has been carried out. The maximum difference, the sum of all absolute differences and the rms value are given in Table 9. The percentage distribution of the differences in the various intervals is shown in the Table 10. The list of the differences which are bigger than 5 $\mu$as in longitude and obliquity is presented in Table 11. These three tables clearly demonstrate that more than 90% of the terms can be computed with comparable accuracy as in other nutation series. However significant differences occur for the largest nutation terms (with periods 18.6 y, 9.3 y, 365 d, 182 d, 13 d) and for the long-periodic nutation terms, say above 18.6 years. Both should be expected due to the reasons explained in Sect. 4.

Next, a comparison between H96NUT and SMART97 in time domain was performed. Thus the nutation angles $\Delta\psi$ and $\Delta\varepsilon$ were evaluated numerically starting at JED = 2 396 931.666 with a time step of 20 hours and a total step number of 131 072 ($2^{17}$) covering approximately 300 years. It follows from Table 9 that the largest difference of about 5 500 $\mu$as at 6 786 days would dominate the comparison in time domain. Therefore, instead of showing the figures in that case another comparison in time domain was carried out where all nutation terms with periods larger than 6 700 days were omitted in order to compare in a better manner the short-periodic terms. Indeed, the differences in time domain drop by one order of magnitude (see Table 12). They are shown for the time interval 1850–2150 in Fig. 2.

## 6. Conclusions

The aim of this paper was to present a new and complete series for the nutation of a rigid Earth model using the geophysical approach. The originality of this work is that a completely independent analytical computation method has been established. Using the most precise tidal potential model HW95 (Hartmann & Wenzel 1995a,b) and a single computation method for all nutation terms results in a precise nutation series H96NUT. All contributions up to 0.45 $\mu$as were taken into account leading to 699 nutation terms. This is an improvement by one order of magnitude in the truncation threshold comparing with the series of KS90. The underlying theory, the main results, some error estimations and a comparison with other recent nutation series were presented. It has been shown that except for the long-periodic and largest nutation terms this computation method can compete with the traditional method and is a useful validation of the existing nutation series. The nutation series H96NUT for the figure axis, the angular momentum axis and the rotation axis together with a standard Fortran-77 or an ANSI-C program to evaluate these series are available at the URL:
`ftp://astro.geo.tu-dresden.de/pub/h96nut`.

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## References


Andoyer H., 1923, 1926, Cours de Mécanique Céleste, Vol. 2. Gauthier-Vilars, Paris

Aoki S., Guinot B., Kaplan G.H., Kinoshita. H., McCarthy D.D., Seidelmann P.K., 1982, A&A 105, 359

Bretagnon P., Francou G., Rocher W., Simon J.L., 1998, A&A 329, 329

Capitaine N., Williams J.G., Seidelmann P.K., 1985, A&A 146, 381

Damour T., Soffel M., Xu C., 1993, Phys. Rev. D 47, 3124

Dehant V., Defraigne P., 1997, J. Geophys. Res. 102(B12), 27 659

Dehant V., Arias F., Brzeziński A., et al., 1997, Report of WG on nutation, Version 7

Fukushima T., 1991, A&A 244, L11

Hartmann T., 1996, Ph.D. thesis, University of Tübingen, Germany

Hartmann T., Soffel M., 1994, AJ 108(3), 1115

Hartmann T., Wenzel H.-G., 1995a, Bull. Inf. Marées Terrestres 123, 9278

Hartmann T., Wenzel H.-G., 1995b, Geophys. Res. Lett. 22(24), 3553

Hartmann T., Soffel M., Kioustelidis T., 1994, Celest. Mech. 60, 139

Hartmann T., Williams J.G., Soffel M., 1996, AJ 111(3), 1400

Herring T.A., 1991, The ZMOA-1990 nutation series. in: Proc. of the 127th Colloquium. of the IAU, Hughes J.A., Smith C.A., Kaplan G.H. (eds.). U. S. Naval Obs., Washington, p. 157

Herring T.A., 1996, in: IERS Conventions, McCarthy D.D. (ed.). Observatoire de Paris, p. 25

Kinoshita H., 1972, PASJ 24, 423

Kinoshita H., 1977, Celest. Mech. 15, 277

Kinoshita H., Souchay J., 1990, Celest. Mech. 48, 187

Lieske J.H., Lederle T., Fricke W., Morando B., 1977, A&A 58, 1

Mathews P.M., Buffet B.A., Herring T.A., Shapiro I.I., 1991a, J. Geophys. Res. 96(B5), 8219

Mathews P.M., Buffet B.A., Herring T.A., Shapiro I.I., 1991b, J. Geophys. Res. 96(B5), 8243

McCarthy D.D., 1992, IERS Standards 1992, IERS Technical Note 13, Observatoire de Paris

Melchior P., Georis N., 1968, Phys. Earth Planet. Int. 1, 267

Roosbeek F., 1996, Geophys. J. Int. 126, 197

Roosbeek F., Dehant V., 1997, Celest. Mech. (submitted)

Seidelmann P.K., 1982, Celest. Mech. 27, 79

Simon J.L., Bretagnon P., Chapront J., Chapront-Touzé M., Francou G., Laskar J., 1994, A&A 282, 663

Souchay J., 1993, A&A 276, 266

Souchay J., Kinoshita H., 1996, A&A 312, 1017

Souchay J., Kinoshita H., 1997a, A&A 318, 639

Souchay J., Kinoshita H., 1997b, Nutation of rigid Earth SK96.2 (priv. comm.)

Souchay J., Loysel B., Kinoshita H., Folgueira M., 1998, A&A (accepted)

Standish E.M., Williams J.G., 1981, Planetary and lunar ephemerides DE200-LE200, magnetic tape

Standish E.M., Newhall X.X., Williams J.G., Folkner W.F., 1995, JPL IOM 314, 10

Vondrák J., 1982, Bull. Astron. Inst. Czech. 33, 26

Vondrák J., 1983a, Bull. Astron. Inst. Czech. 34, 184

Vondrák J., 1983b, Bull. Astron. Inst. Czech. 34, 311

Wahr J.M., 1981, Geophys. J. R. Astron. Soc. 64, 705

Wenzel H.-G., 1996, Bull. Inf. Marées Terrestres 124, 9394

Williams J.G., 1994, AJ 108(2), 711

Williams J.G., 1995, AJ 110(3), 1420

Woolard E.W., 1953, Astron. Papers Am. Ephem. 15, Part 1

Wünsch J., 1991, Astron. Nachr. 312(5), 321

Zhu S.Y., Groten E., 1989, AJ 98(3), 1104