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Legendre expansion of the $\nu\bar{\nu} \rightleftharpoons e^+e^-$ kernel: Influence of high order terms

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Abstract. We calculate the Legendre expansion of the rate of the process $\nu + \bar{\nu} \leftrightarrow e^+ + e^-$ up to 3[rd] order extending previous results of other authors which only consider the 0[th] and 1[st] order terms. Using different closure relations for the moment equations of the radiative transfer equation we discuss the physical implications of taking into account quadratic and cubic terms on the energy deposition outside the neutrinosphere in a simplified model. The main conclusion is that 2[nd] order is necessary in the semi-transparent region and gives good results if an appropriate closure relation is used.

Key words: radiation mechanisms: thermal — radiative transfer — stars: neutron — supernovae: general — nuclear reactions

1. Introduction

The neutrino emission processes play an important role in different astrophysical scenarios, in particular during the stellar core collapse and the cooling of a protoneutron star. Recently, several authors have studied some processes such as neutrino-electron scattering (Smit et al. 1996 and Cernohorsky J. 1994), neutrino coherent scattering off nuclei (Leinson 1992), effects of nucleon spin fluctuations in the weak interaction rates (Janka et al. 1996) or neutrino reactions with strange matter (Reddy & Prakash 1997) in order to obtain the rates to be used in transport calculations. We will focus on the thermal pair emission-absorption process $e^+e^- \leftrightarrow \nu\bar{\nu}$ (TP in the next). In the stellar core collapse scenario, discrepancies in the efficiency of TP in the heating of matter outside the neutrinosphere have been noticed (see, e.g., Janka 1990 and references therein) and in order to obtain meaningful calculations of the energy deposition rate it is necessary to carefully consider the angular dependence of the distribution function.

In some calculations involving neutrino transport, only the 0[th] and 1[st] terms of the Legendre expansion of the collision kernel are included as in Bruenn (1985) and Suzuki (1989). This approach could suffice where the diffusion approximation remains valid. The justification of that approach relies on the fact that the 2[nd] and 3[rd] order terms appear multiplying 2[nd] and 3[rd] order terms of the Legendre expansion of the neutrino distribution function (I) and they vanish in the diffusion limit. However, if a general closure relation is used which is different to $P = \frac{1}{3}E$, these higher order terms of I do not vanish, and their contribution becomes especially important in the semi-transparent region. In this paper we will also analyze the influence of the different closure relations that have been considered in recent years.

This work is organized as follows: In Sect. 2 we present the Legendre expansion of the TP production and absorption kernels and we give explicit expressions for the 0[th] to 3[rd] order terms. In Sect. 3 we study how these new terms affect the sources in the two-moment closure transport equations. In Sect. 4 we discuss the effects of the new terms and the influence of the closure relation.

2. Legendre expansion of emission-absorption TP kernels

Following Bruenn (1985) the contribution of thermal pair production and absorption of neutrino–antineutrino pairs to the right hand side of the Boltzmann equation (TP collision term) is

$$B_{\rm TP}[I,\bar{I}] = \frac{1}{c(hc)^3}\int_0^\infty \omega'^2 {\rm d}\omega' \int_{-1}^{+1} {\rm d}\mu' \int_0^{2\pi} {\rm d}\phi$$
$$\left\{[1-I][1-\bar{I}]R^p_{\rm TP}(\omega,\omega',\cos\theta) - I\bar{I}R^a_{\rm TP}(\omega,\omega',\cos\theta)\right\} \quad (1)$$

where $I = I(t,r,\mu,\omega)$ ($\bar{I} = \bar{I}(t,r,\mu',\omega')$) is the neutrino (antineutrino) invariant distribution function, ω (ω') is the neutrino (antineutrino) energy in the frame comoving with

Send offprint requests to: J.A. Pons, Jose.A.Pons@uv.es the matter, μ (μ') the cosine of the angle between the neutrino (antineutrino) momentum and the polar axis, ϕ is the azimutal angle, and θ is the angle between neutrino and antineutrino directions. In what follows, the explicit dependence on t and r of the distribution functions will be omitted and we will assume axial symmetry with respect to the polar axis. The superscripts a and p refer to absorption and production, respectively.

Assuming electrons and positrons in equilibrium at a temperature T, the following relation between the absorption and production kernels is satisfied

$$R^a_{\mathrm{TP}}(\omega, \omega', \cos\theta) = \mathrm{e}^{\frac{\omega+\omega'}{T}} R^p_{\mathrm{TP}}(\omega, \omega', \cos\theta) \quad (2)$$

where the temperature is measured in units of energy. Expanding R^p_{TP} as follows

$$R^p_{\mathrm{TP}}(\omega, \omega', \cos\theta) = \sum_l \frac{2l+1}{2} \Phi_l(\omega, \omega') P_l(\cos\theta) \quad (3)$$

where $P_l(\cos\theta)$ are the Legendre polynomials and $\Phi_l(\omega, \omega')$ are the Legendre moments of the production kernel, the TP collision term can be written as

$$B_{\mathrm{TP}}[I, \bar{I}] = \frac{2\pi}{c(hc)^3} \int_0^\infty \omega'^2 \mathrm{d}\omega'$$
$$\left\{ (1-I)\Phi_0 - \sum_l \frac{2l+1}{2} \Phi_l P_l(\mu) \int_{-1}^{+1} \mathrm{d}\mu' P_l(\mu') \bar{I} + \right.$$
$$\left. \left(1 - \mathrm{e}^{\frac{\omega+\omega'}{T}}\right) \sum_l \frac{2l+1}{2} \Phi_l P_l(\mu) I \int_{-1}^{+1} \mathrm{d}\mu' P_l(\mu') \bar{I} \right\}. \quad (4)$$

In order to obtain the expressions for Φ_l, we need to evaluate the following integral over electron energy E

$$\Phi_l = \frac{G^2}{\pi} \int_0^{\omega+\omega'} \mathrm{d}E F_{\mathrm{e}}(E, \eta) F_{\mathrm{e}}(\omega + \omega' - E, -\eta) \times$$
$$\left[\alpha_1^2 J_l(\omega, \omega', E) + \alpha_2^2 J_l(\omega', \omega, E)\right] \quad (5)$$

where α_1 and α_2 are summarized in Table 1 for the different neutrino types and $F_{\mathrm{e}}(E, \eta)$ is the Fermi-Dirac distribution function

$$F_{\mathrm{e}}(E, \eta) = \frac{1}{\mathrm{e}^{\frac{E}{T} - \eta} + 1} \quad (6)$$

η being the electron degeneracy parameter, and

$$J_l(\omega, \omega', E) = \frac{1}{\omega\omega'} \Theta(\omega + \omega' - E) \times$$
$$\int_{-1}^{+1} \mathrm{d}\mu P_l(\mu) \left[A(\mu) + B(\mu)E + C(\mu)E^2\right] \Theta(\mu - \mu_0) \quad (7)$$

$$\mu_0 = 1 - \frac{2E(\omega + \omega' - E)}{\omega\omega'}. \quad (8)$$

In the previous expression Θ stands for the step function.

Although $A(\mu)$, $B(\mu)$ and $C(\mu)$ are complicated functions of μ, ω and ω', the integrals J_l can be done analytically, and the results for $l = 0, 1$ were first obtained by Bruenn (1985). We have calculated the integrals for $l = 2, 3$ and their dependence on E is simply a polynomial law of degree $2l + 5$ with coefficients being functions of ω and ω'.

Table 1. Coefficients α_i for different neutrino species

	$\nu_{\mathrm{e}}\bar{\nu}_{\mathrm{e}}$	$\nu_{\mu,\tau}\bar{\nu}_{\mu,\tau}$
α_1	$1 + 2\sin^2\theta_w$	$-1 + 2\sin^2\theta_w$
α_2	$2\sin^2\theta_w$	$2\sin^2\theta_w$

Then, taking into account the following relation

$$F_{\mathrm{e}}(E, \eta) F_{\mathrm{e}}(\omega + \omega' - E, -\eta) =$$
$$\frac{1}{1 - \mathrm{e}^{\frac{\omega+\omega'}{T}}} \left[F_{\mathrm{e}}(E, \eta) - F_{\mathrm{e}}\left(E, \eta + \frac{\omega+\omega'}{T}\right)\right] \quad (9)$$

the Φ_l functions can be expressed in a simple way in terms of the dimensionless variables $y = \omega/T$ and $z = \omega'/T$.

$$\Phi_l(y, z) = \frac{G^2}{\pi} \frac{T^2}{1 - \mathrm{e}^{(y+z)}} \left[\alpha_1 \Psi_l(y, z) + \alpha_2 \Psi_l(z, y)\right] \quad (10)$$

$$\Psi_l(y, z) = \sum_{n=0}^{2} \left(c_{ln} G_n(y, y+z) + d_{ln} G_n(z, y+z)\right)$$
$$+ \sum_{n=3}^{2l+5} a_{ln} \left(G_n(0, \min(y, z)) - G_n(\max(y, z), y+z)\right) \quad (11)$$

where

$$G_n(a, b) = \int_a^b \mathrm{d}x \frac{x^n}{\mathrm{e}^{x-\eta} + 1} - \int_a^b \mathrm{d}x \frac{x^n}{\mathrm{e}^{x-(\eta+y+z)} + 1}\cdot \quad (12)$$

The explicit expressions for a_{ln}, c_{ln} and d_{ln} coefficients are in Appendix A and the method to evaluate the $G_n(a, b)$ integrals is detailed in Appendix B.

3. Energy and momentum transfer in two moment neutrino transport

Two moment neutrino transport (Cernohorsky & van Weert 1992) consists in solving the spectral energy and momentum balance equations as a coupled set. Let us define the l^{th} moment of the distribution function

$$I_l(\omega) = \frac{1}{2} \int_{-1}^{+1} I \mu^l \mathrm{d}\mu \quad (13)$$

and the following *Eddington factors*

$$f(\omega) = \frac{I_1(\omega)}{I_0(\omega)} \tag{14}$$

$$p(\omega) = \frac{I_2(\omega)}{I_0(\omega)} \tag{15}$$

$$q(\omega) = \frac{I_3(\omega)}{I_0(\omega)}. \tag{16}$$

For the sake of simplicity we will omit the energy dependence. Quantities with bar ($\bar{I}$) will stand for antineutrinos and analogous definitions to (13-16) will be used.

In order to close the set formed by the two equations (energy and momentum) we need two closure relations $p = p(f, I_0)$ and $q = q(f, I_0)$. In the next section we give some closure relations widely used in the literature.

The source terms in the energy and momentum balance equations are obtained by angular integration of the collision term on the right hand side of the Boltzmann equation

$$\left(\frac{\partial I_0}{\partial t}\right)_{\mathrm{TP}} = \frac{1}{2}\int_{-1}^{1} \mathrm{d}\mu B_{\mathrm{TP}}$$
$$= \frac{2\pi}{c(hc)^3}\int_0^\infty \omega'^2\,\mathrm{d}\omega'\,S_0(\omega,\omega') \tag{17}$$

$$\left(\frac{\partial I_1}{\partial t}\right)_{\mathrm{TP}} = \frac{1}{2}\int_{-1}^{1} \mathrm{d}\mu\mu B_{\mathrm{TP}}$$
$$= \frac{2\pi}{c(hc)^3}\int_0^\infty \omega'^2\,\mathrm{d}\omega'\,S_1(\omega,\omega'). \tag{18}$$

These source terms are the contribution of the TP process to the energy and momentum exchange between neutrinos and matter. After using the Legendre expansion (4) the expression for S_0 and S_1 can be obtained

$$S_0 = \left[1 - I_0 - \bar{I}_0 + \left(1 - \mathrm{e}^{\frac{\omega+\omega'}{T}}\right)\Gamma_0(\omega,\omega')\right]\Phi_0 \tag{19}$$

$$S_1 = -I_1\Phi_0 - \bar{I}_1\Phi_1 + \left(1 - \mathrm{e}^{\frac{\omega+\omega'}{T}}\right)\Gamma_1(\omega,\omega') \tag{20}$$

$$\Gamma_0 = \sum_l \frac{2l+1}{4}\frac{\Phi_l}{\Phi_0}\int_{-1}^{1}\mathrm{d}\mu P_l(\mu)\,I\int_{-1}^{1}\mathrm{d}\mu' P_l(\mu')\,\bar{I} \tag{21}$$

$$\Gamma_1 = \sum_l \frac{2l+1}{4}\Phi_l\int_{-1}^{1}\mathrm{d}\mu P_l(\mu)\,\mu\,I\int_{-1}^{1}\mathrm{d}\mu' P_l(\mu')\,\bar{I}. \tag{22}$$

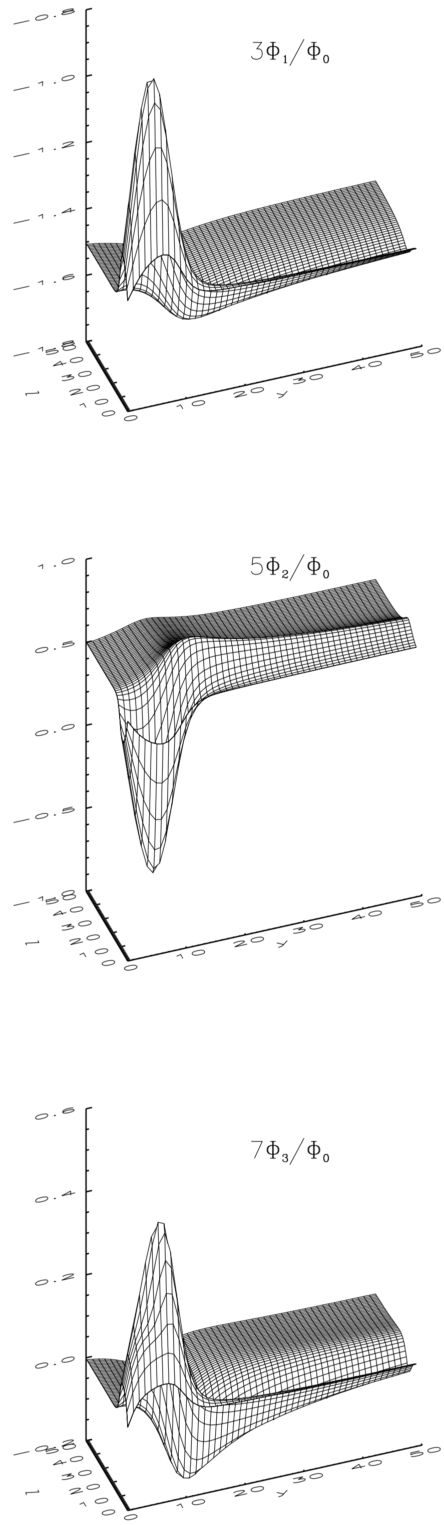

Fig. 1. Ratios $(2i{+}1)\Phi_i/\Phi_0$ for $i = 1, 2, 3$ as functions of $y = \frac{\omega}{T}$ and $z = \frac{\omega'}{T}$ for an electron degeneracy parameter $\eta_{\mathrm{e}} = 10$ and μ–τ neutrino type

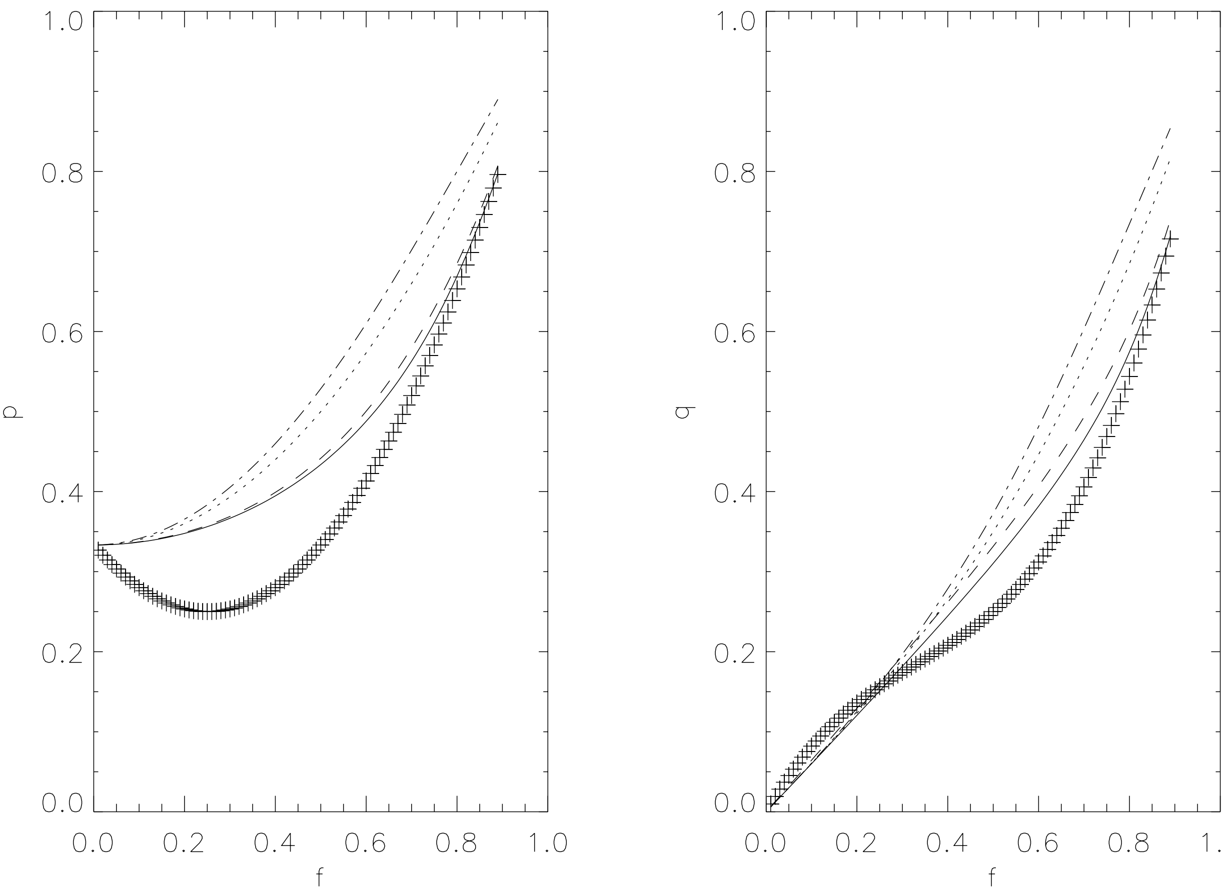

Fig. 2. Plots of $p(f)$ and $q(f)$ for different closure relations. The solid line is for CB closure, dotted line for MH, dashed line for MB and dashed-dotted line for LP. We also plot with crosses the closure obtained in the vacuum approximation

4. Discussion

If neutrino transport calculations are done in the diffusion approximation, where the distribution function is truncated at first order

$$I(\mu,\omega) = I_0(\omega) + 3\mu I_1(\omega) \tag{23}$$

the expansion in (21) is also truncated at the same order and, hence, there is no need to calculate high order terms of the kernels. However, in the semitransparent region where the diffusion approximation breaks down and the flux factor is big, one needs to use a closure relation which is different from $P = \frac{1}{3}E$ and the terms in Φ_l for $l \geq 2$ must be taken into account because, as we will demonstrate, keeping only the first order term would give a wrong value of the energy exchange between neutrinos and matter. To see this fact more clearly, let us study the influence on the energy source of 2$^{\rm nd}$ and 3$^{\rm rd}$ order terms truncating $\Gamma_0(\omega,\omega')$ at $l = 3$.

$$\Gamma_0(\omega,\omega') = 1 + \{f\bar{f}\}\, 3\frac{\Phi_1}{\Phi_0} + \left\{\frac{(3p-1)}{2}\frac{(3\bar{p}-1)}{2}\right\} 5\frac{\Phi_2}{\Phi_0} + \left\{\frac{(5q-3f)}{2}\frac{(5\bar{q}-3\bar{f})}{2}\right\} 7\frac{\Phi_3}{\Phi_0}. \tag{24}$$

In this expression we have written inside brackets {} the part of the correction due to the distribution function. The value of these factors is restricted to the interval $[0,1]$. Therefore, the maximum value of each new term is $(2l+1)\frac{\Phi_l}{\Phi_0}$, which we have plotted in Figs. 1a,b and c for $l = 1, 2, 3$, respectively. As we can see from the plots, if terms in brackets are not small, the 2$^{\rm nd}$ contribution should be included for all energies and the 3$^{\rm rd}$ order term has a significant contribution for low energies. Only when the terms in brackets are much smaller than 1 the expansion can be truncated at first order.

The closure relation becomes, therefore, a fundamental point in the calculation of the $e^+e^- \leftrightarrow \nu\bar{\nu}$ emission-absorption rate. There are several closure relations used by different authors and the question "which is the best one?" has no answer yet. For the sake of comparison, in this work we include four different closure relations: MB (Minerbo 1978), LP (Levermore & Pomraining 1981, CB (Cernohorsky & Bludman 1994) and MH (Mihalas 1984). Cernohorsky closure depends on both, the 0$^{\rm th}$ and 1$^{\rm st}$ moments of the distribution function $p = p(I_0, f)$ and

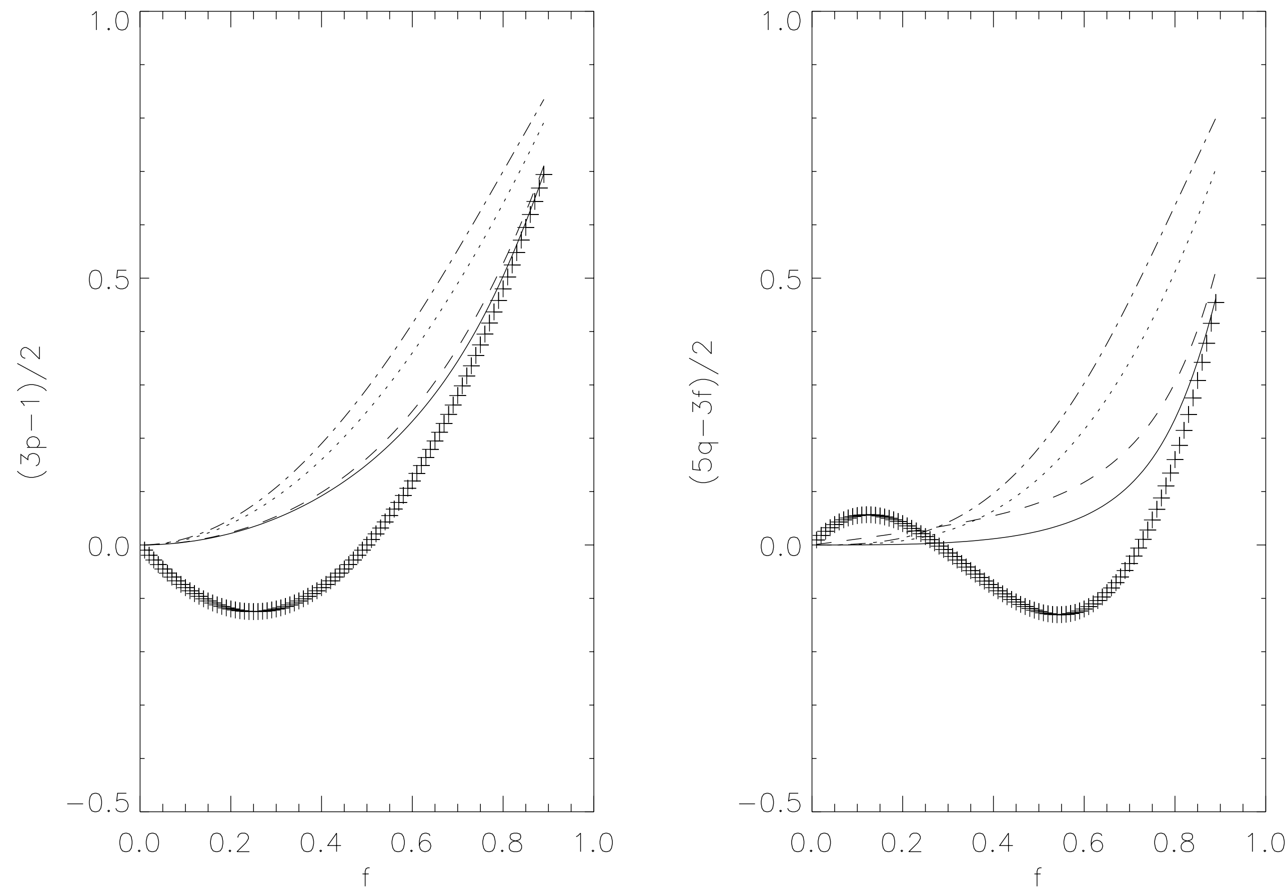

Fig. 3. Terms in brackets in the 2nd and 3rd order contributions in the Legendre expansion for different closures. The meaning of the lines is the same as in Fig. 2

$q = q(I_0, f)$ and the rest of them are uniparametric closures, this is, $p = p(f)$ and $q = q(f)$. In the small occupation limit ($I_0 \ll 1$) CB closure is equivalent to Minerbo's one and in the maximal forward angular packing limit is equivalent to the *vacuum approximation* closure (see below).

The form of these closures is the following:

$$MB \begin{cases} f(a) = \coth(a) - 1/a \\ p(a) = 1 - 2f(a)/a \\ q(a) = f(a) - (3p(a) - 1)/a \end{cases} \tag{25}$$

$$LP \begin{cases} f(a) = \coth(a) - 1/a \\ p(a) = \coth(a) f(a) \\ q(a) = \coth(a) p(a) - 1/3a \end{cases} \tag{26}$$

$$MH \begin{cases} p(f) = (1 + 2f^2)/3 \\ q(f) = (3f + 2f^3)/5 \end{cases} \tag{27}$$

$$CB \begin{cases} p(I_0, f) = \frac{1}{3} + \frac{2}{3}(1 - I_0)(1 - 2I_0)\chi\left(\frac{f}{1-I_0}\right) \\ q(I_0, f) = \text{non-analytic} \end{cases} \tag{28}$$

where $\chi(x) = 1 - 3x/\beta(x)$ with $\beta(x)$ being the inverse of the Langevin function $x = \coth\beta - 1/\beta$. The following expression fits this function well (Cernohorsky & Bludman 1994).

$$\chi(x) = \frac{x^2(3 - x + 3x^2)}{5}.$$

In Fig. 2 we show $p(f)$ and $q(f)$ for the different closure relations, taking an occupation level for the CB case of $I_0 = 0.1$. In Fig. 3, the combinations of the different moments that appear inside brackets in (24) are plotted as a function of f. In both figures the solid line is for CB closure, dotted line for MH, dashed line for MB and dashed-dotted line for LP. We also plot with crosses the closure obtained in the vacuum approximation. As can be seen from the figures, there are important differences between different closures for $f \geq 0.3$. This can lead to relevant differences in the energy exchange between neutrinos and matter for large values of the flux factor f.

To illustrate this feature, let us study a simple model consisting of a sphere of radius R radiating neutrinos and antineutrinos isotropically into vacuum, the so called *vacuum approximation* (Cooperstein et al. 1986). The closure

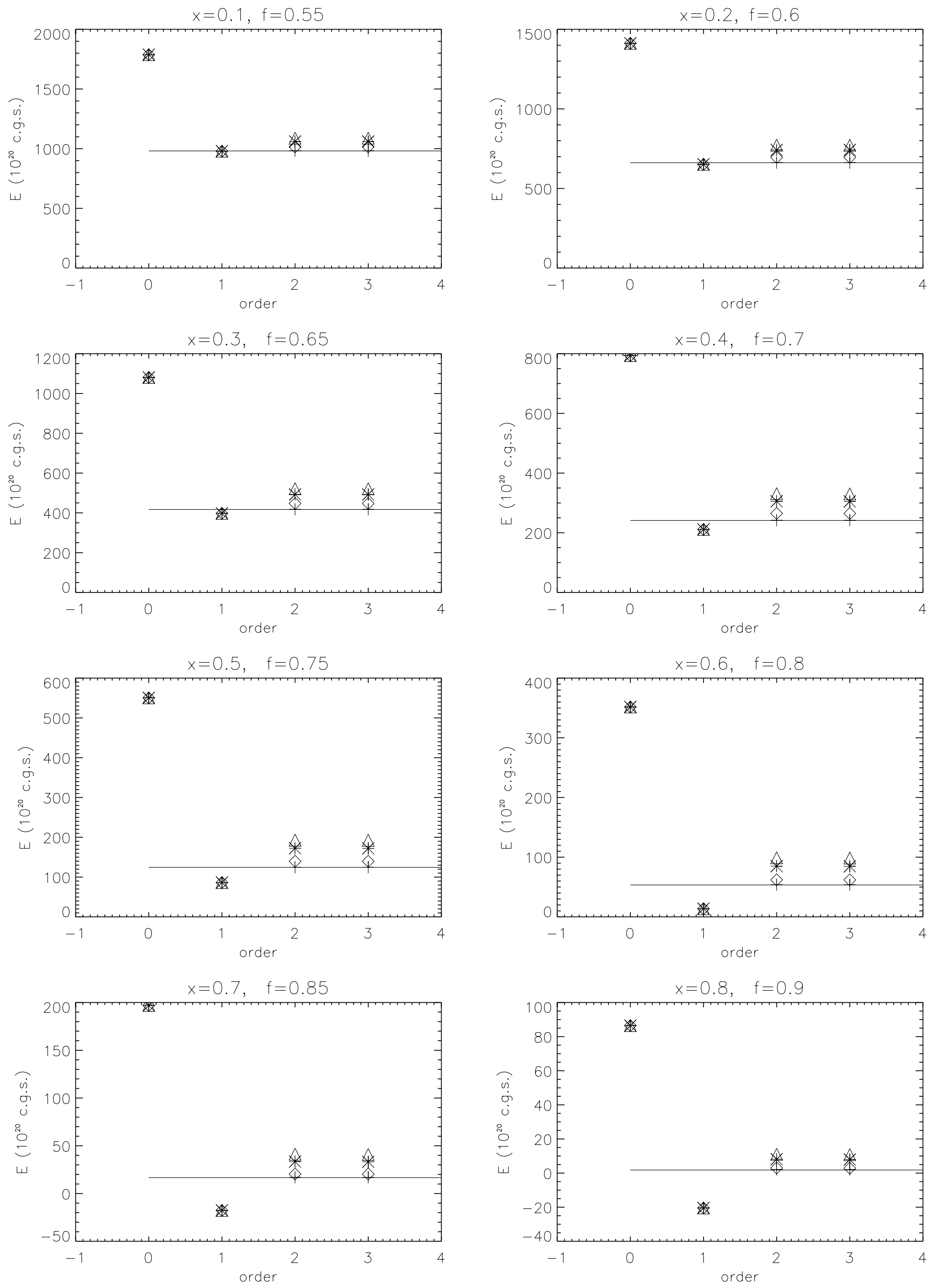

Fig. 4. Total energy deposited for different values of $x = \sqrt{1-(R/d)^2}$ in units of 10^{20} erg cm^{-3} s^{-1}. The solid line is the exact solution after numerical integration and the different symbols stand for the closures LP (triangle), MH (star), MB (diamond) and CB (crosses)

consistent with this model (VA) is

$$VA \begin{cases} f(a) = \dfrac{1+a}{2} \\ p(a) = \dfrac{1+a+a^2}{3} \\ q(a) = \dfrac{1+a+a^2+a^3}{4}. \end{cases} \tag{29}$$

We assume that the neutrino (and antineutrino) spectrum at the surface of the sphere is Fermi-Dirac with zero chemical potential and $T_\nu = 1$ MeV. Of course in a real case the neutrinospheres of neutrinos and antineutrinos are located in different places for different energies, but this example is illustrative of the general behaviour of the heating rate. We calculate the net (heating minus cooling) heating rate of the matter per unit volume for given distance to the center of the sphere (d) and matter temperatures (T). For $T > T_\nu$ cooling dominates over heating and we find that the effect of including new orders is negligible. In contrast, for low T the dominant term in Eq. (19) is the term proportional to Γ_0 and there are remarkable differences in the heating rates obtained including the new terms. These effects are also closure dependent and we compare the closure relations discussed previously in order to estimate the differences between them.

In Fig. 4 we present the result of our calculations. We fix the matter temperature at $T = 0.5$ MeV and we study the influence of higher order terms and closures for different distances from the center of the sphere (different flux factors). For the sake of comparison, we have performed Monte Carlo integration of the complete expression for the reaction rate, shown in the figure as the solid line. We overplot the total energy deposition after including each new order, using different symbols for the different closures. As can be seen, at first order there is an underestimation of the deposition rate that is worse for high flux factors, even changing the sign (this means emission instead absorption of energy) for $f > 0.8$. The inclusion of the second order term gives a big improvement if one uses the closure consistent with the form of the distribution function in the model. The third order term is a small correction that can be omitted in all cases unless high accuracy is required. Using different closures instead of VA gives worse results at small x, but solves the problem in the sign for high values of x. We also observe remarkable differences between the results obtained using different closures, even though, we cannot deduce which one would have better behaviour in a realistic case.

To summarize, we restate our main conclusions. The first conclusion is that, in the semi-transparent region and for matter temperature lower than neutrino temperature, it is necessary to consider the expansion up to 2$^{\rm nd}$ order. This procedure gives good results when combined with an appropriate closure relation. The 3$^{\rm rd}$ order term does not lead to a substantial improvement in the solution since, as we have shown, it is only a small correction. For $T > T_\nu$ cooling dominates over heating and we find that the effect of including new orders is negligible.

The second conclusion is that, even though convergence is reached with 2$^{\rm nd}$ order corrections, the result is very sensitive to the closure relation chosen. Best results are obtained when using a closure relation consistent with the particular distribution function used in the model and therefore, detailed study of the closure in each particular problem is needed to obtain good estimates of interaction rates in a multigroup flux-limited diffusion problem. These results can be applied in all problems concerning neutrino transport such as stellar core collapse or cooling of newly born neutron stars.

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Appendix A: Expressions of a_{ln}, c_{ln} and d_{ln} coefficients

The coefficients for $l = 0, 1$ are the same as in Bruenn (1985) but adapted to our notation.

Coefficients for $l = 0$

$$c_{00} = \frac{4y}{z^2}\left(\frac{2z^2}{3} + yz + \frac{2y^2}{5}\right)$$

$$d_{00} = \frac{4z^3}{15y^2}$$

$$c_{01} = -\frac{4y}{3z^2}(3y + 4z)$$

$$d_{01} = -\frac{4z^2}{3y^2}$$

$$c_{02} = \frac{8y}{3z^2}$$

$$d_{02} = \frac{8z}{3y^2}$$

$$a_{03} = \frac{8}{3y^2}$$

$$a_{04} = -\frac{4}{3y^2 z}$$

$$a_{05} = \frac{4}{15y^2 z^2}.$$

Coefficients for $l = 1$

$$c_{10} = -\frac{4y}{z^3}\left(\frac{2}{7}y^3 + \frac{4}{5}y^2 z + \frac{4}{5}yz^2 + \frac{1}{3}z^3\right)$$

$$d_{10} = -\frac{4z^3}{105y^3}(14y + 9z)$$

$$c_{11} = \frac{4y}{z^3}\left(\frac{4}{5}y^2 + \frac{7}{5}yz + \frac{2}{3}z^2\right)$$

$$d_{11} = \frac{4z^2}{5y^3}\left(\frac{7}{3}y + 2z\right)$$

$$c_{12} = -\frac{4y}{z^3}\left(\frac{3}{5}y + \frac{1}{3}z\right)$$

$$d_{12} = -\frac{4z}{y^3}\left(\frac{1}{3}y + \frac{3}{5}z\right)$$

$$a_{13} = \frac{8}{3y^2}$$

$$a_{14} = -\frac{4}{3y^3z}(4y + 3z)$$

$$a_{15} = \frac{4}{15y^3z^2}(13y + 18z)$$

$$a_{16} = -\frac{4}{5y^3z^3}(y + 3z)$$

$$a_{17} = \frac{16}{35y^3z^3}.$$

Coefficients for $l = 2$

$$c_{20} = \frac{4y}{z^4}\left(\frac{2}{7}y^4 + \frac{6}{7}y^3z + \frac{32}{35}y^2z^2 + \frac{2}{5}yz^3 + \frac{1}{15}z^4\right)$$

$$d_{20} = \frac{4z^3}{105y^4}\left(16y^2 + 27yz + 12z^2\right)$$

$$c_{21} = -\frac{4y}{z^4}\left(\frac{6}{7}y^3 + \frac{12}{7}y^2z + yz^2 + \frac{2}{15}z^3\right)$$

$$d_{21} = -\frac{4z^2}{y^4}\left(\frac{18}{35}z^2 + \frac{6}{7}yz + \frac{1}{3}y^2\right)$$

$$c_{22} = \frac{8y}{5z^4}\left(\frac{1}{6}z^2 + \frac{3}{2}yz + \frac{12}{7}y^2\right)$$

$$d_{22} = \frac{8z}{5y^4}\left(\frac{1}{6}y^2 + \frac{3}{2}yz + \frac{12}{7}z^2\right)$$

$$a_{23} = \frac{8}{3y^2}$$

$$a_{24} = -\frac{4}{3y^3z}(10y + 9z)$$

$$a_{25} = \frac{4}{15y^4z^2}(73y^2 + 126yz + 36z^2)$$

$$a_{26} = -\frac{12}{y^4z^3}(y^2 + 3yz + \frac{8}{5}z^2)$$

$$a_{27} = \frac{48}{35y^4z^4}(2y^2 + 13yz + 12z^2)$$

$$a_{28} = -\frac{24}{7y^4z^4}(y + 2z)$$

$$a_{29} = \frac{8}{7y^4z^4}.$$

Coefficients for $l = 3$

$$c_{30} = -\frac{4y^2}{z^5}\left(\frac{10}{33}y^4 + \frac{20}{21}y^3z + \frac{68}{63}y^2z^2 + \frac{18}{35}yz^3 + \frac{3}{35}z^4\right)$$

$$d_{30} = -\frac{4z^3}{105y^5}\left(9y^3 + 34y^2z + 40yz^2 + \frac{500}{33}z^3\right)$$

$$c_{31} = \frac{4y^2}{z^5}\left(\frac{20}{21}y^3 + \frac{130}{63}y^2z + \frac{48}{35}yz^2 + \frac{9}{35}z^3\right)$$

$$d_{31} = \frac{4z^2}{35y^5}\left(3y^3 + 24y^2z + \frac{130}{3}yz^2 + \frac{200}{9}z^3\right)$$

$$c_{32} = -\frac{4y^2}{z^5}\left(\frac{50}{63}y^2 + \frac{6}{7}yz + \frac{6}{35}z^2\right)$$

$$d_{32} = -\frac{4z^2}{y^5}\left(\frac{50}{63}z^2 + \frac{6}{7}zy + \frac{6}{35}y^2\right)$$

$$a_{33} = \frac{8}{3y^2}$$

$$a_{34} = -\frac{4}{3y^3z}(19y + 18z)$$

$$a_{35} = \frac{4}{15y^4z^2}(253y^2 + 468yz + 180z^2)$$

$$a_{36} = -\frac{8}{15y^5z^3}(149y^3 + 447y^2z + 330yz^2 + 50z^3)$$

$$a_{37} = \frac{8}{21y^5z^4}\left(116y^3 + \frac{2916}{5}y^2z + 696yz^2 + 200z^3\right)$$

$$a_{38} = -\frac{40}{21y^5z^5}(5y^3 + 54y^2z + 108yz^2 + 50z^3)$$

$$a_{39} = \frac{40}{21y^5z^5}\left(10y^2 + 43yz + \frac{100}{3}z^2\right)$$

$$a_{3(10)} = -\frac{40}{9y^5z^5}(3y + 5z)$$

$$a_{3(11)} = \frac{320}{99y^5z^5}.$$

Appendix B: Evaluation of the G_n integrals

The integrals $G_n(a,b)$ appearing in the expression of the Legendre moments can easily be expressed as sums or differences of the Fermi-like integral

$$F_n(\eta, x_1) = \int_0^{x_1} \mathrm{d}x \frac{x^n}{\mathrm{e}^{x-\eta}+1}$$

and therefore, the problem is reduced to calculate this kind of integrals.

In order to do this, we expand the denominator in the previous expression (Sack 1990), which must be done in a different way depending on the sign of $a = x - \eta$, this is:

$$\frac{1}{\mathrm{e}^a+1} = \begin{cases} \sum_{m=0}^{\infty} (-1)^m \mathrm{e}^{ma} & a<0 \\ \sum_{m=0}^{\infty} (-1)^m \mathrm{e}^{-(m+1)a} & a>0 \end{cases}$$

then, we obtain an infinite sum of integrals that can be calculated analytically using

$$\int x^k \mathrm{e}^{mx} = k! \sum_{l=0}^{k} \frac{(-1)^{k-l}}{l!} \frac{\mathrm{e}^{mx} x^l}{m^{k+1-l}}$$

$$\int x^k \mathrm{e}^{-mx} = -k! \sum_{l=0}^{k} \frac{1}{l!} \frac{\mathrm{e}^{-mx} x^l}{m^{k+1-l}}.$$

Let us define the following function

$$T_l(\alpha) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \mathrm{e}^{-n\alpha}}{n^l}$$

which is well defined for $\alpha > 0$ and $l \geq 1$, we finally arrive to a useful expression for the $F_k(\eta, x_1)$ integrals depending on the value of η.

If $\eta < 0$

$$F_k(\eta, x_1) = k! \left[T_{k+1}(-\eta) - \sum_{l=0}^{k} \frac{T_{k+1-l}(x_1-\eta) x_1^l}{l!} \right].$$

If $0 \leq \eta \leq x_1$

$$F_k(\eta, x_1) = \frac{\eta^{k+1}}{k+1} + k! \left[2 \sum_{l=0}^{\mathrm{INT}\left[\frac{k-1}{2}\right]} \frac{T_{2l+2}(0) \eta^{k-1-2l}}{(k-1-2l)!} + (-1)^k T_{k+1}(\eta) - \sum_{l=0}^{k} \frac{T_{k+1-l}(x_1-\eta) x_1^l}{l!} \right].$$

If $x_1 < \eta$

$$F_k(\eta, x_1) = \frac{x_1^{k+1}}{k+1} + k! \left[(-1)^k T_{k+1}(\eta) - \sum_{l=0}^{k} (-1)^{k-l} \frac{T_{k+1-l}(\eta - x_1) x_1^l}{l!} \right].$$

The previous expressions are exact, and we only have to calculate a sum of a finite number of terms (up to k). The accuracy depends exclusively on the evaluation of the $T_l(\alpha)$ functions. The fact that these are uniparametric functions allows us to tabulate them in a fine grid at the beginning of the calculation and to obtain enough accuracy without excessive CPU time cost.

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