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Method of running parabolae: Spectral and statistical properties of the smoothing function

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Abstract. Analytic expressions in a general form are derived for the expectation of functions related to the smoothing the astronomical signals using local approximations with additional weights. No restrictions are made on distribution of the times of the signal. Applications are made for polynomial fits of orders 0 and 2 and weights $p(z) = 1$ and $(1-z^2)^2$. These variable weights ensure that the smoothing function will be both continuous and differentiable, which is important for determining the extrema and the shape of the variations. Special attention is paid to evenly spaced time series data, which if their number is large enough will allow one to obtain analytic expressions for the main functions.

Key words: methods: analytical; data analysis; numerical; statistical

1. Introduction

Methods of local approximations are widely used for smoothing the signals of different nature (see classic textbook by Whittaker & Robinson (1928) and e.g. a recent monograph by Hardle (1990) and papers by Foster (1996a,b). We point our attention towards a particular class of smoothing functions and discuss their properties by approximating harmonic signals and the "white noise".

The method of "running parabolae" for smoothing signals with both equidistantly and not equidistantly distributed in time signals was proposed in our Paper I (Andronov 1990) and was applied to light curves of stars of different types, e.g. HQ And (Andronov et al. 1992a), MV Lyr (Andronov et al. 1992b), TT Ari (Tremko et al. 1996), UV Aur, TX CVn, V 1329 Cyg (Chinarova et al. 1994). The advantage of this method as compared with, e.g. smoothing by a running mean is a smooth approximating curve, which has continuous first derivative and a better amplitude–frequency dependence. This allows application of the method to aperiodic and cyclical processes and, particularly, to determine extrema. In this work we analyse properties of the smoothing function in more detail, comparing 4 modifications, namely:

1) "um", unweighted mean, usually referred to as a "running mean" or "moving average";
2) "wm", weighted mean, with weights (5);
3) "up", unweighted parabolae with constant weight;
4) "wp", weighted parabolae with weights (5), called "running parabolae" in Paper I.

Hereafter in the text we will use two-letter abbreviations, whereas in figures - the numerical ones.

2. Basic equations

Let x_k be values of the signal obtained at times t_k, $k = 1...N$. In the "local" (or "running") approximations it is usually suggested that the data $(t_k,\ x_k)$ $(t_0 - \Delta t \le t_k \le t_0 + \Delta t)$ are fitted by a function $\varphi(t, t_0, \Delta t)$, which depends not only on the moment t, but on the limits of the interval of fitting. Examples of such function fitting of a test signal by various methods are shown in Fig. 1. However, the resulting function is expected to be dependent only on one argument. Thus it is generally chosen so that the smoothing ("computed") value x_c at the moment t_0 is equal to a value of the smoothing function $\varphi(t, t_0, \Delta t)$ at $t = t_0$:

$$x_c(t_0) = \varphi(t_0, t_0, \Delta t). \qquad (1)$$

(cf. Whittaker & Robinson 1928). In this case Δt remains a free parameter which determines statistical and spectral properties of the function $x_c(t)$ for a fixed set of data $(t_k,\ x_k)$. Here we assumed that the data are renumerated according to the trial argument interval from $t_0 - \Delta t$ to $t_0 + \Delta t$. Obviously, such numbering is dependent both on the "mean argument" t_0 and on the "filter half-width" Δt.

In the most often case of the linear fits, the function φ may be expressed as

$$\varphi(t, t_0, \Delta t) = \sum_{\alpha=0}^{m} C_\alpha(t_0, \Delta t)\ f_\alpha(t - t_0), \qquad (2)$$

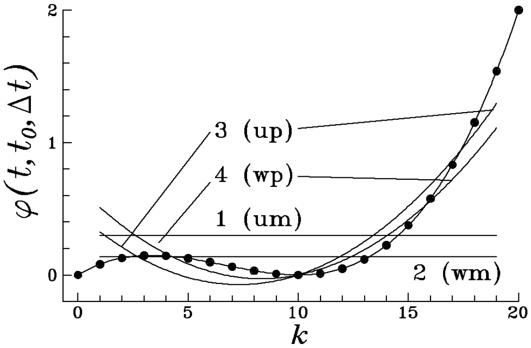

Fig. 1. Approximations $\varphi(t, t_0, \Delta t)$ of the model discrete "signal" $x_k = t_k^3 + t_k^2$ defined at times $t_k = k$, $(k = 0...20)$ by using 4 tested fits for $t_0 = 10$ and $\Delta t = 10$ ("wm", "wp") or $\Delta t = 9$ ("um", "up"). Such difference in Δt leads to equal number $n = 19$ of observations with non–zero weights. A smoothing value of φ at $t = t_0$ corresponds to an adopted value $x_{\rm c}(t_0)$

where the coefficients C_α may be determined e.g. by minimizing a weighted sum of the residuals

$$\Phi(t_0, \Delta t) = \sum_{k=1}^{n} p_k\ w_k\ (x_k - \varphi(t_k, t_0, \Delta t))^2. \tag{3}$$

The weights w_k are generally characteristic of the accuracy σ_k of the measurements x_k and are equal to $w_k = \sigma_0^2/\sigma_k^2$, where σ_0 is an "unit weight error", if $p_k = 1$ for the data used for the fit (cf. Whittaker & Robinson 1928). The parameter σ_0 is a scale coefficient which may be set to arbitrary positive value. It does not affect the smoothing function and its statistical characteristics. The "additional" weights $p_k = p((t_k - t_0)/\Delta t)$ were used in Paper I to make the smoothing function and its first derivative continuous.

The following concrete functions were used:

$$p(z) = \begin{cases} 1 & \text{if } |z| \le 1 \\ 0 & \text{else} \end{cases} \tag{4}$$

("unweighted" fits), and

$$p(z) = \begin{cases} (1 - z^2)^2 & \text{if } |z| < 1 \\ 0 & \text{else} \end{cases} \tag{5}$$

("weighted" fits). As the base functions, we have used the polynomials:

$$f_\alpha(z) = z^\alpha. \tag{6}$$

The minimum of the function $\Phi(t_0, \Delta t)$ for the fixed data corresponds to a system of "normal" equations:

$$\sum_{\alpha=0}^{m} A_{\alpha\beta}\ C_\alpha = \sum_{k=1}^{n} p_k\ w_k\ x_k\ f_\beta(t_k - t_0), \tag{7}$$

where

$$A_{\alpha\beta} = \sum_{k=1}^{n} p_k\ w_k\ f_\alpha(t_k - t_0)\ f_\beta(t_k - t_0). \tag{8}$$

Introducing a vector of values

$$h[C_\alpha, k] = p_k\ w_k \sum_{\beta=0}^{m} A_{\alpha\beta}^{-1}\ f_\beta(t_k - t_0), \tag{9}$$

one may write

$$C_\alpha = \sum_{k=1}^{n} h[C_\alpha, k]\ x_k. \tag{10}$$

In our designations, $A_{\alpha\beta}^{-1}$ is a matrix, inverse of $A_{\alpha\beta}$, and

$$x(t_0) = \sum_{\alpha=0}^{m} C_\alpha(t_0, \Delta t)\ f_\alpha(0). \tag{11}$$

This means that coefficients C_α and basic functions $f_\alpha(t)$ have "interchanged their places": C_α are now functions of t_0 and Δt, whereas values of the basic functions are constant.

Introducing a vector $h[x_{\rm c}, k]$ similar to (9), one may write

$$h[x_{\rm c}, k] = p_k\ w_k \sum_{\alpha\beta=0}^{m} A_{\alpha\beta}^{-1}\ f_\beta(t_k - t_0)\ f_\alpha(0). \tag{12}$$

This vector is also a function of t_0, Δt. Each $k-$th component of this vector may be interpreted as a dependence of the calculated value $x_{\rm c}(t_0)$ smoothing the unit value $x_k = 1$, whereas all other signal values are equal to zero. For N evenly distributed observations $t_i = t_1 + (i-1)\Delta$ $(i = 1...N)$ with a step Δ, the function h of 3 variables $(t_k, t_0, \Delta t)$ becomes dependent on 2 variables $((t_k - t_0), \Delta t)$ only. In this case, one may write a convolution – type expression

$$x_{\rm c}(t_i) = \sum_{k=1}^{n} h[x_{\rm c}, k]\ x(t_{i+k-k'}). \tag{13}$$

Here k' is a number corresponding to $t_{k'} = t_0$ in each interval of the local approximation. This equation is valid for $i = k'...N - n + k'$. For the "borders" $(i = 1...k' - 1, N - n + k' + 1...N)$ one has to redetermine the vector $h[x_{\rm c}, k]$. In Paper I we determined values of h for the illustrative 9-point "wp" fits.

In this paper we prefer to express all fit functions X (coefficients, derivatives etc.) in terms of the "projective" vectors $h[X, k]$, because this allows one to estimate the accuracy and possible correlations between the parameters.

If δX and δY are deviations of the functions X and Y caused by deviations of the observations δx_k, then

$$\delta X\ \delta Y\ =\ \sum_{i,k=1}^{n} \frac{\partial X}{\partial x_i}\,\frac{\partial Y}{\partial x_k}\ \delta x_i\ \delta x_k \qquad (14)$$
$$=\ \sum_{i,k=1}^{n}\ h[X,i]\ h[Y,k]\ \delta x_i\ \delta x_k.$$

Mathematical expectation of the left side of this equation may be calculated, if the correlation matrix $\delta x_i\ \delta x_k$ or its mathematical expectation $\langle \delta x_i\ \delta x_k\rangle$ is known.

For uncorrelated deviations $\langle\delta x_i\rangle\ =\ 0$, $\langle\delta x_i\ \delta x_k\rangle\ =\ \sigma_i^2\ \delta_{ik}$, where δ_{ik} is a Kronecker symbol, and Eq. (14) may be rewritten as

$$\langle\delta X\ \delta Y\rangle\ =\sum_{k=1}^{n}\ h[X,k]\ h[Y,k]\ \sigma_k^2. \qquad (15)$$

In the particular case $Y = X$ one obtains variance of X: $\sigma^2[X] =\ \langle\delta X\ \delta X\rangle$. For the coefficients C_α one may obtain relation

$$\langle\delta C_\alpha\ \delta C_\beta\rangle\ =\sigma_0^2\ R_{\alpha\beta}, \qquad (16)$$

where

$$R_{\alpha\beta} = \sum_{\gamma,\epsilon=0}^{m}\ A^{-1}_{\alpha\gamma}\ G_{\gamma\epsilon}\ A^{-1}_{\epsilon\beta}, \qquad (17)$$

and

$$G_{\gamma\epsilon} = \sum_{k=1}^{n}\ p_k^2\ w_k\ f_\gamma(t_k-t_0)\ f_\epsilon(t_k-t_0). \qquad (18)$$

One may note that for $p_k\ =\ p\ =$ const, the matrices $G_{\gamma\epsilon}\ =\ pA_{\gamma\epsilon}$, $R_{\alpha\beta}\ =\ pA^{-1}_{\alpha\beta}$. For the unweighted fits one usually suggests $p = 1$, thus $G_{\gamma\epsilon} = A_{\gamma\epsilon}$, $R_{\alpha\beta} = A^{-1}_{\alpha\beta}$. This last result is usual for least squares approximations (cf. Whittaker & Robinson 1928; Anderson 1958).

Following Paper I, one may formally separate "true" (index "t") values of the signal $x_{\rm t\it k}$ and the deviations of the real observations from them $x_{{\rm d}k} = x_k - x_{{\rm t}k}$. The values $x_{{\rm d}k}$ are often believed to be uncorrelated with each other and the "true" values, have a zero mathematical expectation and a variance $\sigma_k^2\ =\ \sigma_0^2/w_k$. Usually "true" values are unknown (except models with known "signal" and "noise"), but may have systematic deviations from the corresponding fit which may be characterized by a parameter $\Phi_{\rm t}$. The weighted sum (3) of the squares of the residuals $(x_k-\varphi(t_k,t_0,\Delta t))$ is

$$\Phi = \Phi_{\rm t} + \left(\sum_{k=1}^{n}\ p_k - \sum_{\alpha\beta=0}^{m}\ A^{-1}_{\alpha\beta}\ G_{\alpha\beta}\right)\ \sigma_0^2 = \Phi_{\rm t}+\psi\sigma_0^2, \qquad (19)$$

where

$$\Phi_{\rm t} = \sum_{k=1}^{n}\ p_k\ w_k\ [x_{{\rm t}k}-\varphi_{\rm t}(t_k,t_0,\Delta t)]^2. \qquad (20)$$

The second summand in the right part of Eq. (19) may be called $\Phi_{\rm d}$ as corresponds to the deviations. Equation (19) allows to estimate of σ_0^2 which is needed for accuracy determinations. One may note that, for constant weights (4), the expression in brackets in Eq. (19) is equal to $(n-m-1)$ for all non-degenerate systems of basic functions $f_{\alpha k}$. Usually the fits are chosen so that $\Phi_{\rm t}$ is negligible as compared with Φ, i.e. it is suggested that the systematic deviation of the fit from the true signal is much less than its statistical error. The values of Φ, $\phi_{\rm d}$, $\Phi_{\rm t}$, ψ and the estimate Φ/ψ of σ_0^2 are dependent on t_0 and Δt.

The variance of the smoothed value at argument t_0 is

$$\sigma^2[x_{\rm c}(t_0)] = \langle(x_{\rm cd}(t_0))^2\rangle\ \ =\ \sum_{\alpha\beta=0}^{m}\ \langle\delta C_\alpha\ \delta C_\beta\rangle\ f_\alpha(0)\ f_\beta(0) \qquad (21)$$
$$=\ \sigma_0^2\ \sum_{\alpha\beta=0}^{m}\ R_{\alpha\beta}\ f_\alpha(0)\ f_\beta(0).$$

Here $x_{\rm cd}(t_0)$ is the deviation of the smoothed value $x_{\rm c}(t_0)$ from the true one $x_{\rm ct}(t_0)$.

If the argument t_0 coincides with t_k of the $k-$th observation, then one may transform Eq. (23) of Paper I into

$$\langle w_k\ (x_{{\rm d}k}-x_{\rm cd}(t_k))^2\rangle = \qquad (22)$$
$$=\ \sigma_0^2\,(1+w_k\sum_{\alpha\beta=0}^{m} f_\alpha(0) f_\beta(0)(R_{\alpha\beta}-2p_kA^{-1}_{\alpha\beta}))$$

For polynomial fits $f_\alpha(0) = \delta_{0\alpha}$ and for the weights (4) and (5) $p_k = 1$, thus $\langle w_k\ (x_{{\rm d}k}-x_{{\rm cd}k})^2\rangle\ =\ \sigma_0^2\ (1+w_k\ (R_{00} - 2\ A^{-1}_{00}))$. For "constant" weights (4), taking into account that in this case $R_{\alpha\beta} = A^{-1}_{\alpha\beta}$, one may obtain even more simple expression $\langle w_k\ (x_{{\rm d}k}-x_{{\rm cd}k})^2\rangle\ =\ \sigma_0^2\ (1-w_k\ A^{-1}_{00})$. Making summation of the left and right parts of Eqs. (22, 19) for all observations (or only part of them), one may also estimate σ_0^2 neglecting systematic deviations of the fit from the true signal as compared with the statistical error of the signal value. These values we will mark as σ_1^2 and σ_2^2. Another characteristic value of the variance may be defined as

$$\sigma_3^2 = \frac{\sum\limits_{k=1}^{n} w_k(x_k-x_{\rm c}(t_k))^2}{\sum\limits_{k=1}^{n} w_k}. \qquad (23)$$

For normally distributed uncorrelated signal the values σ_1 and σ_2 must be very close as they characterize the same quantity – the unbiased estimate of the unit weight variance. The parameter σ_3 is the rms deviation of the signal from the fit, its mathematical expectation depends on Δt and is biased.

One may note that general expressions for the smoothed value and its accuracy may cause problems, if the number of points n_1 in the subinterval $[t_0-\Delta t, t_0+\Delta t]$ is not sufficient. If $n_1 = m+1$ and all the arguments t_k are different then one may obtain the fit interpolating all the values. If number of different arguments is smaller than $m+1$, the system of normal equations is degenerate and no fit of order m is available. In this case one may decrease m (what changes statistical and spectral properties of the fit) or not to use the fit at this data point. We prefer the second way when computing σ_1, σ_2 and σ_3.

Computation of the smoothed values at the moments of observations is carried out most often to compute estimates of σ_0^2 and to provide time series analysis of the residuals O$-$C of the signal from the fit. Another application is to compute the fit at arbitrary argument t_0. For this case we propose to use the following restrictions: a) the number of the data points inside the interval must exceed $m+1$ (as was mentioned above); b) the numbers of the data points with $t_k < t_0$ (j_1) and $t_k > t_0$ (j_2) must both be nonzero; c) the number of the data points with $|t_k - t_0| \le \alpha\Delta t$ (j_3) must exceed some limiting value (practically $j_3 \ge 3$ and $\alpha = 0.3$); d) the accuracy estimate of the smoothed value $\sigma[x_{\rm c}(t_0)]$ must not exceed some limiting value, e.g. σ_1 or manually inserted one; e) the value $x_{\rm c}(t_0)$ must lie within the interval $[(1+\beta)x_{\rm min} - \beta x_{\rm max}, (1+\beta)x_{\rm max} - \beta x_{\rm min}]$, where one may recommend to use β from 0 to 0.1. These restrictions (some of them may be not used) allow to obtain the fit only at arguments t_0 where it has sense, because in other case one may formally obtain values extrapolating the data at the edge(s) of the subinterval and apparent waves which are not statistically significant.

This may be more simple, if the observations are evenly sampled, and the coefficients $h[x,k]$ are the same for all intervals (except edges), as well as the matrices $R_{\alpha\beta}$, $A^{-1}_{\alpha\beta}$, $G_{\alpha\beta}$.

Generally one may introduce 2 scale factors $\sigma_0^2 \to \lambda\sigma_0^2$, $p_k \to \mu p_k$. Then $A_{\alpha\beta} \to \lambda\mu A_{\alpha\beta}$, $A^{-1}_{\alpha\beta} \to \lambda^{-1}\mu^{-1}A^{-1}_{\alpha\beta}$, $G_{\alpha\beta} \to \lambda\mu^2 G_{\alpha\beta}$, $R_{\alpha\beta} \to \lambda^{-1}R_{\alpha\beta}$, $\Phi \to \lambda\mu\Phi$, but the parameters $h[C_\alpha,k]$, C_α, $\varphi(t,t_0,\Delta t)$, $x_{\rm c}(t_0)$, $\sigma_0^2 R_{\alpha\beta}$ do not depend on λ and μ, thus they may be set to any nonzero value. Practically one may choose $\mu = 1$ and $w_k = \sigma_k^{-2}$ (i.e. $\sigma_0^2 = 1$) for unequal weights and $w_k = 1$ for equal weights. It is important to note that generally the parameter σ_0^2 itself does not correspond to the characteristics accuracy of the observations. The accuracy of the fit is defined in a more complex way by Eq. (21).

Foster (1996c) proposed to introduce the parameter

$$\sigma_m^2 = \sigma_0^2 \frac{\sum_{k=1}^n p_k^2 w_k}{\left[\sum_{k=1}^n p_k w_k\right]^2}. \qquad (24)$$

This quantity is scale–invariant, as does not depend on parameters λ and μ. It has physical sense of the variance of the parameter

$$x_m = \frac{\sum_{k=1}^n p_k w_k x_k}{\sum_{k=1}^n p_k w_k} \qquad (25)$$

which coincides with a weighted mean. Imposing the normalization

$$\sum_{k=1}^n w_k = n = \sum_{k=1}^n \frac{\sigma_0^2}{\sigma_k^2}, \qquad (26)$$

one may define the "local" ensemble variance

$$\sigma_*^2 = \frac{n_*}{\sum_{k=n_1}^{n_2} \sigma_k^{-2}} = \frac{\sigma_0^2}{\bar{w}}, \qquad \bar{w} = \frac{1}{n_*}\sum_{k=n_1}^{n_2} w_k, \qquad (27)$$

where $n_* = n_2 - n_1 + 1$ is the number of the data points (from n_1 to n_2) in the trial interval $[t_0 - \Delta t, t_0 + \Delta t]$. In previous expressions the sums from 1 to n and from n_1 to n_2 were equal as they contained the additional weight p_k which is equal to zero for k outside the interval $[n_1, n_2]$. The use of points in a local interval is correct from the statistical point of view but is not suitable to use the ensemble variance which vary with t_0. Thus one may introduce the "global" ensemble variance which is defined for the whole data set and does not depend on the interval $[t_0 - \Delta t, t_0 + \Delta t]$:

$$\bar{w}_n = \frac{1}{n}\sum_{k=1}^n w_k. \qquad (28)$$

Foster (1996c) proposes to use σ_*^2 instead of σ_0^2 as it has clear physical meaning. In other notation, this corresponds to $\lambda = 1/\bar{w}$. From Eqs. (23) and (25) one may define the effective number of data points

$$n_{\rm eff} = \frac{\left[\sum_{k=1}^n p_k w_k\right]^2}{\bar{w}\sum_{k=1}^n p_k^2 w_k}. \qquad (29)$$

With a normalization $\bar{w} = 1$ one will obtain $n_{\rm eff}$ in a form by Foster (1996c). One may note that $n_{\rm eff} = n_*$ for equal weights $p_k = p$, and $n_{\rm eff} < n_*$ for unequal weights.

In our notation, these expressions are meaningful for "um" and "wm", as x_m coincides with the smoothing value at t_0. For parabolic and other non–linear fits one may redefine the effective number of data points using $h[x_{\rm c},k]/w_k$ instead of p_k in Eqs. (23, 24, 27):

$$n_{\rm eff} = \frac{\sigma_*^2}{\sigma^2[x_{\rm c}(t_0)]} = \frac{1}{\bar{w}\sum_{\alpha\beta=0}^m R_{\alpha\beta} f_\alpha(0) f_\beta(0)}. \qquad (30)$$

3. Derivatives of the smoothing function

According to definition (1), the smoothing function $x_c(t)$ coincides with the function $\varphi(t, t_0.\Delta t)$ at points $t = t_0$. However, this in not the case for the derivatives, i.e.

$$\frac{\partial^s}{\partial t_0^s}\, x_c(t_0) = \left(\frac{\partial^s}{\partial t_0^s}\, \varphi(t_0, t_0, \Delta t)\right)_{t_0} \neq$$
$$\neq \left(\frac{\partial^s}{\partial t^s}\, \varphi(t, t_0, \Delta t)\right)_{t=t_0} = \sum_{\alpha=0}^{m} C_\alpha \left(\frac{\partial^s}{\partial t^s} f_\alpha(t)\right)_{t=0}. \tag{31}$$

Obviously, for the $s-$th derivative of a general parameter X,

$$h[\partial^s X/\partial t_0^s, k] = \frac{\partial^s h[X, k]}{\partial t_0^s}. \tag{32}$$

Equation (31) allows to estimate accuracy $\sigma[\partial^s\varphi/\partial t^s]$ of the $s-$th derivative $\partial^s\varphi(t, t_0, \Delta t)/\partial t^s$:

$$\sigma^2\left[\frac{\partial^s}{\partial t^s}\varphi(t, t_0, \Delta t)\right] = \tag{33}$$
$$= \sigma_0^2 \sum_{\alpha\beta=1}^{m} R_{\alpha\beta} \left(\frac{\partial^s}{\partial t^s} f_\alpha(t - t_0)\right)\left(\frac{\partial^s}{\partial t^s} f_\beta(t - t_0)\right).$$

Particularly, if $t = t_0$ and polynomial basic functions (6),

$$\frac{\partial^s}{\partial t^s}\phi(t, t_0, \Delta t) \;=\; s!\; C_s, \tag{34}$$

and thus

$$\sigma^2\left[\frac{\partial^s}{\partial t^s}\phi(t, t_0, \Delta t)\right] \;=\; \sigma_0^2\;(s!)^2\; R_{ss}. \tag{35}$$

Much more complicated is the determination of the derivatives by the argument t_0. Most important are first and second derivatives, especially at the extrema. Let's determine the vectors h at a moment $t_0 + \delta$. For small δ, one may expand vectors into series restricting maximum order to 2:

$$A_{\alpha\beta} = A_{\alpha\beta 0} \;+\; A_{\alpha\beta 1}\,\delta \;+\; A_{\alpha\beta 2}\,\delta^2 \;+\; \ldots \tag{36}$$

And similarly for $A^{-1}_{\alpha\beta}$, C_α, h_k and $f_{\alpha k} = f_\alpha(t_k - t_0)$. Hereafter the last index $s = 0, 1, 2$ corresponds to a coefficient at δ^s. For the coefficient C_0:

$$h[C_{00}, k] = w_k \sum_{\beta=0}^{m} A^{-1}_{0\beta 0}\, p_{k0}\, f_{\beta k0} \tag{37}$$

$$h[C_{01}, k] = w_k \sum_{\beta=0}^{m} [A^{-1}_{0\beta 0}\, p_{k0}\, f_{\beta k1} + A^{-1}_{0\beta 0}\, p_{k1}\, f_{\beta k0} \tag{38}$$
$$+ A^{-1}_{0\beta 1}\, p_{k0}\, f_{\beta k0}]$$

$$h[C_{02}, k] = w_k \sum_{\beta=0}^{m} A^{-1}_{0\beta 0}(p_{k0}\, f_{\beta k2} + p_{k1}\, f_{\beta k1} + p_{k2}\, f_{\beta k0})]$$
$$+ \sum_{\beta=0}^{m} [A^{-1}_{0\beta 1}\,(p_{k0}\, f_{\beta k1} + p_{k1}\, f_{\beta k0}) + A^{-1}_{0\beta 2}\, p_{k0}\, f_{\beta k0}]. \tag{39}$$

The coefficients of the power series (35) for parameter X may be written as $X_s = (-1)^s/s!{\cdot}(\partial^s X/\partial t^s)_{t=t_0}$, as all differences $(t - t_0)$ become $(t - t_0 - \delta)$. However, for $A^{-1}_{\alpha\beta}$ it is more suitable to use the expressions

$$A^{-1}_{\alpha\beta 0} \;=\; (A_{\alpha\beta 0})^{-1}, \tag{40}$$
$$A^{-1}_{\alpha\beta 1} \;=\; -A^{-1}_{\alpha\gamma 0}\; A_{\gamma\epsilon 1}\; A^{-1}_{\epsilon\beta 0}, \tag{41}$$
$$A^{-1}_{\alpha\beta 2} \;=\; -A^{-1}_{\alpha\gamma 0}\; (A_{\gamma\epsilon 1}\; A^{-1}_{\epsilon\beta 1} + A_{\gamma\epsilon 2}\; A^{-1}_{\epsilon\beta 0}). \tag{42}$$

For further study of the polynomial fits, we will measure times in units of Δt, practically using dimensionless units $z = (t - t_0)/\Delta t$. Introducing the sums $S_s = \sum_{k=1}^{n} z_k^s$, one may easily obtain

$$A_{\alpha\beta 0} = \begin{pmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{pmatrix}, \quad A_{\alpha\beta 1} = \begin{pmatrix} 0 & -S_0 & -2S_1 \\ -S_0 & -2S_1 & -3S_2 \\ -2S_1 & -3S_2 & -4S_3 \end{pmatrix}, \tag{43}$$
$$A_{\alpha\beta 2} = \begin{pmatrix} 0 & 0 & S_0 \\ 0 & S_0 & 3S_1 \\ S_0 & 3S_1 & 6S_2 \end{pmatrix}$$

for "unweighted" parabolic fits, and, for the weights (5) the matrices $A_{\alpha\beta 0}$, $A_{\alpha\beta 1}$, $A_{\alpha\beta 2}$ are equal to

$$\begin{pmatrix} S_0 - 2S_2 + S_4 & S_1 - 2S_3 + S_5 & S_2 - 2S_4 + S_6 \\ S_1 - 2S_3 + S_5 & S_2 - 2S_4 + S_6 & S_3 - 2S_5 + S_7 \\ S_2 - 2S_4 + S_6 & S_3 - 2S_5 + S_7 & S_4 - 2S_6 + S_8 \end{pmatrix},$$
$$\begin{pmatrix} 4S_1 - 4S_3 & -S_0 + 6S_2 - 5S_4 & -2S_1 + 8S_3 - 6S_5 \\ -S_0 + 6S_2 - 5S_4 & -2S_1 + 8S_3 - 6S_5 & -3S_2 + 10S_4 - 7S_6 \\ -2S_1 + 8S_3 - 6S_5 & -3S_2 + 10S_4 - 7S_6 & -4S_3 + 12S_5 - 8S_7 \end{pmatrix},$$
$$\begin{pmatrix} -2S_0 + 6S_2 & -6S_1 + 10S_3 & S_0 - 12S_2 + 15S_4 \\ -6S_1 + 10S_3 & S_0 - 12S_2 + 15S_4 & 3S_1 - 20S_3 + 21S_5 \\ S_0 - 12S_2 + 15S_4 & 3S_1 - 20S_3 + 21S_5 & 6S_2 - 30S_4 + 28S_6 \end{pmatrix}. \tag{44}$$

Other matrices $A^{-1}_{\alpha\beta s}$, $G_{\alpha\beta}$, $R_{\alpha\beta}$ may be consequently determined by using above mentioned expressions.

To determine extremum of the smoothing function, one has to solve an equation

$$\frac{\partial \dot{x}_c(t_0)}{\partial t_0} = 0. \tag{45}$$

After determining the root t_0 for the given signal values x_k, one has to obtain an accuracy estimate of it. Assuming small variations δt of the moment of the extremum caused by small $\delta\dot{x}_k$, one may write a linearized equation

$$\sum_{k=1}^{n} [h[C_{\alpha 1}, k]\;\delta\dot{x}_k + 2h[C_{\alpha 2}, k]\;\ddot{x}_k\;\delta t] = 0, \tag{46}$$

or

$$\delta t = -\frac{\delta\dot{x}_c}{\ddot{x}_c}, \qquad \sigma[t] = \frac{\sigma[\dot{x}_c]}{|\ddot{x}_c|} \tag{47}$$

where the first ($\dot{x}_c = x_{c1}$) and second ($\ddot{x}_c = 2x_{c2}$) derivatives of the smoothing function x_c are evaluated at argument t_0. Assuming again $\langle \delta x_k \rangle = 0$, $\langle \delta x_k \, \delta x_i \rangle = \sigma_k^2 \, \delta_{ki}$, one may obtain

$$\sigma^2[\dot{x}_c] = \sigma_0^2 \sum_{k=1}^{n} h[x_{c1}, k]^2. \tag{48}$$

As an illustration of the derived above expressions, we show in Fig. 2 the dependence of $h[C_*, k]$ on k for 19-point "wp" approximation.

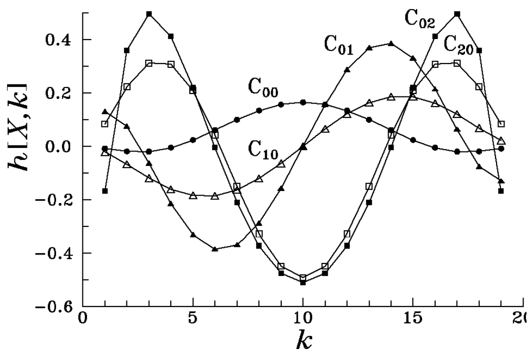

Fig. 2. Projective 19-point vectors $h[C_*, k]$ for polynomial "wp" fits

Statistical properties of the test functions used for the period determination by using the moments of "characteristic events" are studied earlier (Andronov 1987, 1991).

4. Evenly sampled data and limits for continuous functions

The mentioned above equations are valid without restrictions on the distribution of the times of observations. However, evenly spaced signals are also often used in astronomy, and they allow to use the same matrices inside the interval, except the edges. For large number of evenly sampled observations, one may replace sums by integrals:

$$\sum_{k=1}^{n} X(z_k) \approx \frac{n}{z_2 - z_1} \int_{z_1}^{z_2} X(z) \, \mathrm{d}z. \tag{49}$$

Corresponding parameters may be computed as integrals

$$X(t) = \int_{z_1}^{z_2} h[X, z] \, X(t+z) \, \mathrm{d}z. \tag{50}$$

Particularly, an expectation of the rms deviation $\sigma[X]$ of the parameter X is

$$\sigma^2[X] = \sigma_0^2 \int_{z_1}^{z_2} [h[X, z]]^2 \, \mathrm{d}z = \frac{z_2 - z_1}{n} \, \sigma_0^2 \, V[X], \tag{51}$$

where $V[X]$ is a value of the integral characterizing variance of X, and σ_0 is an "unit weight" error (cf. Whittaker & Robinson 1928). Hereafter one may omit a constant $n/(z_2 - z_1)$, when replacing sums by integrals, while not specially mentioned. For example, Eq. (9) may be rewritten as

$$h[C_\alpha, z] = \sum_{\beta=0}^{m} p(z) \, A_{\alpha\beta}^{-1} \, f_\beta(z) \tag{52}$$

For an interval $(-1,\ 1)$ covered by the observations, $S_s = 2/(s+1)$ for even s and 0 else. For constant weights $p(z) = 1$ (case "up") the matrices are the following:

$$A_{\alpha\beta} = \begin{pmatrix} 2 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} \end{pmatrix}, \quad A_{\alpha\beta}^{-1} = \begin{pmatrix} \frac{9}{8} & 0 & -\frac{15}{8} \\ 0 & \frac{3}{2} & 0 \\ -\frac{15}{8} & 0 & \frac{45}{8} \end{pmatrix}. \tag{53}$$

For the coefficients C_α one may obtain

$$\begin{aligned} h[C_0, z] &= \frac{9 - 15z^2}{8}, & V &= \frac{9}{8} \\ h[C_1, z] &= \frac{3z}{2}, & V &= \frac{3}{2}, \\ h[C_2, z] &= \frac{15(3z^2 - 1)}{8}, & V &= \frac{45}{8}. \end{aligned} \tag{54}$$

For simple running mean ("um") $h[C_0, z] = 1/2$, $V = 1/2$, and other coefficients $C_\alpha = 0$ by definition. One may note that coefficients C_1 and C_2 for "unweighted" polynomial fits are equal to $\dot{x}_c$ and $\ddot{x}_c$ (in dimensionless argument units z). However, with changing interval, the observational points are added to and removed from the set, thus such an approximation is valid only for a fixed set of data (t_k, x_k). For continuous functions, one may not choose an interval of t_0 with a fixed set, and derivatives are computed using Eq. (31).

For the weights $p(z) = (1 - z^2)^2$,

$$\begin{aligned} A_{\alpha\beta} &= \begin{pmatrix} \frac{16}{15} & 0 & \frac{16}{105} \\ 0 & \frac{16}{105} & 0 \\ \frac{16}{105} & 0 & \frac{16}{315} \end{pmatrix}, \\ A_{\alpha\beta}^{-1} &= \begin{pmatrix} \frac{105}{64} & 0 & -\frac{315}{64} \\ 0 & \frac{105}{16} & 0 \\ -\frac{315}{64} & 0 & \frac{2205}{64} \end{pmatrix}. \end{aligned} \tag{55}$$

For "weighted mean" ("wm"):

$$h[C_0, z] = \frac{15}{16} \, (1 - z^2)^2, \qquad V = \frac{5}{7}. \tag{56}$$

For "weighted parabolae" ("wp"):

$$h[C_0, z] = \frac{105}{64} \, (1 - z^2)^2 \, (1 - 3z^2), \qquad V = \frac{805}{572}, \tag{57}$$

$$h[C_1, z] = \frac{105}{64} \, (1 - z^2)^2 \, z, \qquad V = \frac{35}{11}, \tag{58}$$

$$h[C_2, z] = \frac{315}{64} \, (1 - z^2)^2 \, (7z^2 - 1), \qquad V = \frac{8505}{572}. \tag{59}$$

To determine $h[C_{01},z]$ and $h[C_{02},z]$, one would use Eqs. (38, 39). However, for continuous polynomial fits and all weight functions satisfying the condition $p(\pm 1)=0$, the matrices $A_{\alpha\beta 1}$, $A_{\alpha\beta 2}$ are equal to zero, as well as $A^{-1}_{\alpha\beta 1}$, $A^{-1}_{\alpha\beta 1}$. For "wp",

$$h[C_{01},z]=\frac{105}{32}\ z\ (1-z^2)\ (5-9z^2),\qquad V=\frac{525}{44},\quad (60)$$

$$h[C_{02},z]=\frac{105}{64}\ (-5+42z^2-45z^4),\qquad V=\frac{2205}{32}.\quad (61)$$

One may note that more complicated character of the evaluation of the derivatives as compared with coefficients C_1 and C_2 leads to much larger values of corresponding parameter V, e.g. $V[C_{01}]/V[C_{10}]=525/140$.

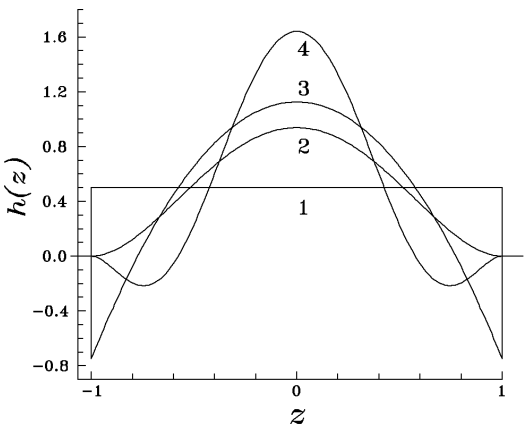

Fig. 3. Projective functions $h[C_0,z]$ for 4 model "symmetric" polynomial fits

First derivative of the smoothing function in the case of continuous signal $x(t)$ is equal to

$$\begin{aligned}\frac{\partial}{\partial t}x_{\rm c}(t) &= \int_{z_1}^{z_2} h[x_{\rm c},z]\ \frac{\partial}{\partial t}x(t+z)\ {\rm d}z = \\ &= h[x_{\rm c},z_2]\ x(t+z_2)-h[x_{\rm c},z_1]\ x(t+z_1) \\ &\quad -\int_{z_1}^{z_2} x(t+z)\ \frac{\partial}{\partial t}h[x_{\rm c},z]\ {\rm d}z.\end{aligned}\qquad (62)$$

For "symmetric" fits ($z_1=-1,\ z_2=1$), one may obtain

$$\frac{\partial}{\partial t}x_{\rm c}(t)\ (\text{"um"}) = \frac{1}{2}(x(t+1)-x(t-1))\qquad (63)$$

$$\frac{\partial}{\partial t}x_{\rm c}(t)\ (\text{"wm"}) = \frac{15}{4}\int_{-1}^{1} z\ (1-z^2)\ x(t+z)\ {\rm d}z\qquad (64)$$

$$\begin{aligned}\frac{\partial}{\partial t}x_{\rm c}(t)\ (\text{"up"}) &= -\frac{3}{4}(x(t+1)-x(t-1)) \\ &+\frac{15}{4}\int_{-1}^{1} z\ x(t+z)\ {\rm d}z\end{aligned}\qquad (65)$$

$$\begin{aligned}\frac{\partial}{\partial t}x_{\rm c}(t)\ (\text{"wp"}) &= \\ &= \frac{105}{32}\int_{-1}^{1} z\ (1-z^2)\ (5-9z^2)\ x(t+z)\ {\rm d}z.\end{aligned}\qquad (66)$$

One may note that derivatives for the "unweighted" fits are strongly dependent on particular values of the signal at the borders, making impossible the form (50) without using the Dirac's $\delta-$ functions. In a case $p(\pm 1)=0$ one may introduce corresponding functions $h[\dot{x}_{\rm c},z]$ and to obtain $V=15/7$ for "wm" and $V=525/22$ for "wp".

Expressions for "asymmetric" fits are much more complicated. For an extreme case $z_1=0,\ z_2=1,\ A_{\alpha\beta}=1/(\alpha+\beta+1)$ for "unweighted" fits, and $A_{\alpha\beta}=8/[(\alpha+\beta+1)(\alpha+\beta+3)(\alpha+\beta+5)]$. Corresponding functions h are equal to 0, if $z<0$ and $z>1$. From 0 to 1 they are the following:

$$h[C_0,z] = 1,\quad V=1\quad (\text{"um"})\qquad (67)$$

$$h[C_0,z] = \frac{15}{8}(1-z^2)^2,\quad V=\frac{10}{7}\quad (\text{"wm"})\qquad (68)$$

$$h[C_0,z] = 9-36z+30z^2,\quad V=9\quad (\text{"up"})\qquad (69)$$

$$\begin{aligned}h[C_0,z] &= \left(\frac{14735}{1224}-\frac{9800}{153}z+\frac{2345}{34}z^2\right)(1-z^2)^2, \\ V &= \frac{36377635}{3347487}\quad (\text{"wp"})\end{aligned}\qquad (70)$$

$$h[C_1,z] = (-36+192z-180z^2),\quad V=192\ (\text{"up"})\qquad (71)$$

$$\begin{aligned}h[C_1,z] &= \left(-\frac{9800}{153}+\frac{71680}{153}z-\frac{9800}{17}z^2\right)(1-z^2)^2, \\ V &= \frac{1111613440}{3347487}\quad (\text{"wp"})\end{aligned}\qquad (72)$$

$$h[C_2,z] = (30-180z+180z^2),\quad V=180\quad (\text{"up"})\qquad (73)$$

$$\begin{aligned}h[C_2,z] &= \left(\frac{2345}{34}-\frac{9800}{17}z+\frac{13230}{17}z^2\right)(1-z^2)^2, \\ V &= \frac{17834460}{41327}\quad (\text{"wp"}).\end{aligned}\qquad (74)$$

5. Fits of the harmonic signals

If the input signal is sinusoidal:

$$x(t) = a\ +\ r\ \cos(2\pi t/P+\phi),\qquad (75)$$

then the corresponding smoothing curve is

$$\begin{aligned}x_{\rm c}(t) &= a+r\,H(\theta)\cos\left(\frac{2\pi t}{P}+\phi\right) \\ &+r\,S(\theta)\sin\left(\frac{2\pi t}{P}+\phi\right),\end{aligned}\qquad (76)$$

where $\theta=2\pi\Delta t/P$ and

$$H(\theta) = +\int_{z_1}^{z_2} h[x_{\rm c},z]\ \cos(\theta z)\ {\rm d}z,\qquad (77)$$

$$U(\theta) = -\int_{z_1}^{z_2} h[x_{\rm c},z]\ \sin(\theta z)\ {\rm d}z.\qquad (78)$$

Asymmetric fits change not only the amplitude, but the phase as well. Such phase distortions occur at the temporal edges of the observations. For intermediate values of the

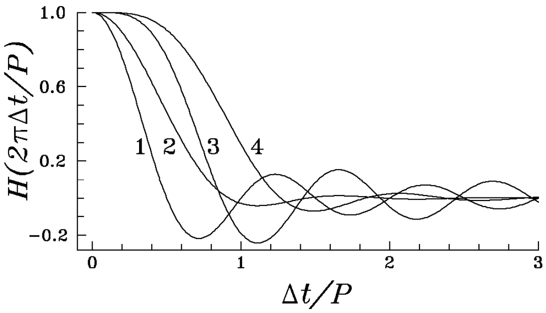

Fig. 4. Dependence of the amplitude of the harmonic fit smoothing a model sine function of unit amplitude and a period P on $\Delta t/P$

argument, $z_1 = -1$ and $z_2 = 1$. Because of the symmetry of the function $h[x_c, -z] = h[x_c, z]$, in this case $U(\theta) = 0$ for all θ.

For our test symmetric fits,

$$H_{\rm um}(\theta) = \frac{\sin\theta}{\theta}, \tag{79}$$

$$H_{\rm wm}(\theta) = \frac{15[(3-\theta^2)\sin\theta - 3\theta\cos\theta]}{\theta^5}, \tag{80}$$

$$H_{\rm up}(\theta) = \frac{3(5-\theta^2)\sin\theta - 15\theta\cos\theta}{2\theta^3}, \tag{81}$$

$$H_{\rm wp}(\theta) = \frac{52.5[(135-57\theta^2+\theta^4)\sin\theta - \theta(135-12\theta^2)\cos\theta]}{\theta^7}. \tag{82}$$

These functions are shown in Fig. 4. A function $H_{\rm wp}(\theta)$ crosses zero at the value $\Delta t = 1.267P$, much larger as compared with $\Delta t = 0.5P$ corresponding to a zero of $H_{\rm um}(\theta)$. Some useful values of $\Delta t/P$ are 0.7047 and 0.8553, which correspond to $H_{\rm wp}(\theta) = 2^{-1/2}$ and 1/2, respectively.

The power spectra $S(x(t), f)$ of the smoothed signal are proportional to r^2 (e.g. Box & Jenkins 1970), thus for harmonic signal with frequency $f = 1/P$ one may obtain $S(x_c(t), f) \approx\ S(x(t), f)\ H^2(2\pi f\Delta t)$. However, for the deviations from the fit

$$S(x(t) - x_c(t), f) \ \approx\ S(x(t), f)\ \rho(\theta), \qquad \rho(\theta)\ =\ (1 - H(2\pi f\Delta t))^2. \tag{83}$$

At high frequencies $H(\theta) \approx 0$, thus fast variations are not affected by removal of a "slow trend" such as the slope γ of the power spectra $S \propto f^{-\gamma}$ detected in some cataclysmic variables (Andronov 1993). Applications corresponding to the integer values $\gamma = 0, 1, 2$ are described by Terebizh (1992). Influence of the trend removal on the shape of the autocorrelation function (ACF) was studied by Andronov (1994).

The functions $H(\theta)$ cross zero for all fits with finite z_1 and z_2. The "ideal" "rectangular" shape $H = 1$, if $f \le f_0$ corresponds to an "ideal low frequency" signal $h[x_c, t] = \sin(2\pi f_0 t)/(\pi t)$ usually described in radiotechnics (e.g. Baskakov 1983). Here $z_1 = -\infty$, $z_2 = \infty$, $\Delta t = 1$. For real functions $h[x_c, z]$ limiting values of z are finite, causing for some frequencies the negative values of $H(\theta)$. In this case, the "output" smoothing function is in anti-phase with the "input" signal, thus the difference $x(t) - x_c(t)$ will have even larger amplitude than the "true" one. The maximum "amplification" occurs for 4 tested methods at $\Delta t/P = 0.715$, 1.112, 1.112, 1.489 with corresponding values of $\rho(\theta)_{\rm max} = 1.4816$, 1.0839, 1.5421, 1.1446. For the "white noise", mathematical expectation of the amplitude does not depend on frequency, thus a power spectrum of the detrended observations may show an apparent peak due to negative values of $H(\theta)$. For symmetric approximations, values of $\rho(\theta)_{\rm max}$ are much smaller for weighted fits, than for unweighted. Case "wp" is not significantly worse than "wm" according to this criterion, but has a strong advantage due to wider shape of $H(\theta)$.

For noisy sinusoidal signal $\Phi_{\rm t} \ll \Phi$ and a "continuous" approximation one may estimate a "signal/noise" ratio

$$\frac{r}{\sigma} = \frac{r_0}{\sigma_+}\sqrt{\frac{n\Delta t}{2V[x_c]}}\ H(2\pi\delta), \tag{84}$$

where $\delta = \Delta t/P$, r_0 is the amplitude of a harmonic component and σ_+ is an accuracy estimate of an individual values which may be set to σ_1. For fixed n (number of observations per period P), this ratio increases with increasing Δt proportionally to $\Delta t^{1/2}$ (as the number of the observations inside an interval increases). However, for large Δt, the amplitude of the smoothed function decreases ($\propto\ H(\theta)$). Thus exists an "optimal" value of $\Delta t/P$, where "signal/noise" reaches its maximum. For continuous approximation, this occurs at $\delta =$ 0.1855, 0.2895, 0.4123, 0.5450 for 4 fits with corresponding $H(\theta) = 0.7885$, 0.7843, 0.8749, 0.8716 and a value $(\Delta t/2V[x_c])^{1/2} H(2\pi\delta) = 0.3396, 0.3531, 0.37450, 0.38353$. This factor decreases by $\sqrt{2}$ times at $\Delta t/P = 0.0608$, 0.0936, 0.1589, 0.2093, i.e. there is a long enough nearly "standstill", if "signal/noise" is plotted vs. δ.

In other words, approximations by a parabola give better results than that by a constant, the "optimal" fit corresponding to "wp" allowing to use more wide intervals than other fits.

Estimates of extremum time errors are possible only for $m > 0$, as

$$\sigma[t_e s] = \frac{\sigma[\dot{x}_c]}{|\ddot{x}_c|}\ \approx\ \sqrt{\frac{2V[\dot{x}_c]}{n\Delta t}}\ \frac{P}{4\pi^2\Delta H(2\pi\delta).} \tag{85}$$

Minimum of this function for fits "up" and "wp" occurs at $\Delta t/P = 0.5515$ and 0.7440 with corresponding $H(2\pi\Delta t/P) = 0.6725$ and 0.65669 and values 0.15929 and 0.2928 $P/n\Delta t^{1/2}$. Here an accuracy estimate for "wp" is worse than for "up". But "up" fit is a discontinuous fit, and sometimes its derivative may be infinite. The accuracy estimate is twice larger than an "optimal" value at $\Delta t/P = 0.2712,\ 0.3612$.

For small noise σ_0 which is comparable with $\sigma_{\rm t}$ and/or discrete signal one may determine numerically the parameter $\Delta t/P$ optimizing a fixed characteristic.

6. Fits of the asymmetric signals

Approximations by a mean or by running parabolae is properly valid, if the signal has a parabolic shape. In real cases, the signal is more complicated. To estimate systematic errors of the fitting curves from the true ones, lets assume that the signal may be expanded into the Taylor series:

$$x(t) = \sum_{m=0}^{\infty} r_m\ (t-t_0)^m, \qquad (86)$$

where

$$r_m = \frac{1}{m!}\ \left(\frac{{\rm d}^m x}{{\rm d}t^m}\right)_{t\,=\,t_0}. \qquad (87)$$

Than the running fit may be written as

$$x_{\rm c}(t_0) = \sum_{m=0}^{\infty} s_m\ \Delta t^m, \qquad (88)$$

where coefficients

$$s_m = \int_{z_1}^{z_2} h[x_{\rm c}, z]\ z^m\ {\rm d}z. \qquad (89)$$

For symmetric fits $s_0 = 1$, and $s_{2k+1} = 0$ for integer k. For even $m = 2k$,

$$s_m\ (\text{"um"}) = \frac{1}{m+1} \qquad (90)$$

$$s_m\ (\text{"wm"}) = \frac{15}{(m+1)(m+3)(m+5)} \qquad (91)$$

$$s_m\ (\text{"up"}) = -\frac{3(m-2)}{2(m+1)(m+3)} \qquad (92)$$

$$s_m\ (\text{"wp"}) = -\frac{105(m-2)}{32(m+1)(m+3)(m+5)(m+7)}. \qquad (93)$$

7. Fits of the signals with abrupt changes of the first derivative

To fit sharp minima of eclipsing variables and asymmetric maxima of pulsating variables, one may use an extreme approximation by a broken line: $x(t)\ =\ a\ u(-t) + b\ u(t)$, where $u(t) = t$, if $t > 0$ and $u(t) = 0$ else. For comparison of the tested methods we used a model

$$x(t)\ =\ u(-t)\ +\ b\ u(t). \qquad (94)$$

Here asymmetry of the extrema is dependent on parameter b. Obviously, one may formally change the sign of time

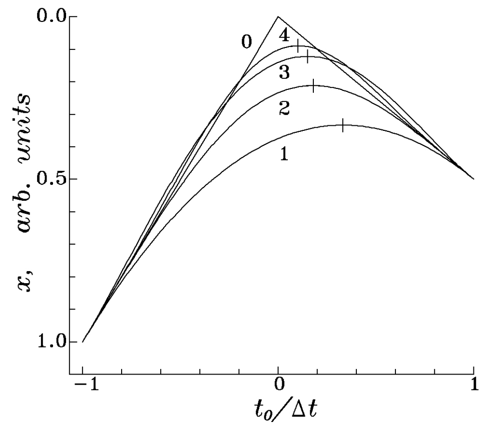

Fig. 5. Approximations of a broken line (0) with $b = 0.5$ by fits (1, 2, 3, 4). The crosses mark extrema of the smoothing functions

to make $b \le 1$. The calculated function coincides with a broken line, if $|t| > 1$ in dimensionless units ($\Delta t = 1$). inside the interval $(-1,\ 1)$

$$x_{\rm c}(t)\ =\ U(-t)\ + b\ U(t), \qquad (95)$$

where

$$U(t) = \int_0^{1+t} \tau\ u(t-\tau)\ {\rm d}\tau. \qquad (96)$$

For 4 methods of smoothing one may obtain $U(t) = (t+1)^2/4$ ("um"), $(t+1)^2(t^4-2t^3-2t^2+6t+5)/32$ ("wm"), $(t+1)^2(-5t^2+10t+3)/32$ ("up") and $-(t+1)^2(45t^6-90t^5-61t^4+212t^3-13t^2-186t-35)/512$ ("wp"), respectively. Dependence on parameter b of times of minima $t_{\rm e}$, smoothed value $x_{\rm c}(t_{\rm e})$ and second derivative $\ddot{x}_{\rm c}(t_{\rm e})$ (last is used in Eq. (47) for determination of the accuracy estimate of the moment of extremum) are presented in Table 1.

As one may see, for a fixed value of Δt, approximation becomes better according to a sequence "um-wm-up-wp". Systematic deviation of the minimum of the smoothing function from the true one is ≈ 3 times smaller for running parabolae ("wp") than for classical running mean ("um"). Unweighted parabolae are shifted ≈ 1.5 times more than "wp".

8. Determination of the optimal value of Δt

The characteristics of the fit are strongly dependent on Δt which is a free parameter. Its determination for concrete data set is a separate problem likewise in the case of determination of the degree of polynomial or the number of harmonics for global approximations. However, in our case the free parameter Δt is continuous and one may not apply the Fischer's statistics to estimate statistical significance of the fit with given Δt.

Table 1. Characteristics of minima of the fits approximating the broken line

b	t_e				$x_c(t_e)$				$\ddot{x}_c(t_e)$			
	um	wm	up	wp	um	wm	up	wp	um	wm	up	wp
0.1	0.8182	0.5247	0.3989	0.2856	0.0909	0.0654	0.0177	0.0155	0.5500	0.5415	0.9093	1.1499
0.2	0.6667	0.3946	0.3134	0.2203	0.1667	0.1128	0.0516	0.0398	0.6000	0.8019	1.1290	1.5229
0.3	0.5385	0.3057	0.2478	0.1724	0.2308	0.1513	0.0793	0.0595	0.6500	1.0016	1.3129	1.8288
0.4	0.4286	0.2373	0.1946	0.1346	0.2857	0.1838	0.1024	0.0758	0.7000	1.1688	1.4756	2.0940
0.5	0.3333	0.1817	0.1500	0.1034	0.3333	0.2119	0.1220	0.0897	0.7500	1.3149	1.6242	2.3313
0.6	0.2500	0.1350	0.1119	0.0769	0.3750	0.2366	0.1389	0.1017	0.8000	1.4459	1.7624	2.5479
0.7	0.1765	0.0947	0.0787	0.0540	0.4118	0.2585	0.1535	0.1122	0.8500	1.5653	1.8928	2.7485
0.8	0.1111	0.0594	0.0494	0.0339	0.4444	0.2783	0.1663	0.1214	0.9000	1.6756	2.0167	2.9362
0.9	0.0526	0.0281	0.0234	0.0160	0.4737	0.2962	0.1775	0.1295	0.9500	1.7784	2.1355	3.1132
1.0	0.0000	0.0000	0.0000	0.0000	0.5000	0.3125	0.1875	0.1367	1.0000	1.8750	2.2500	3.2813

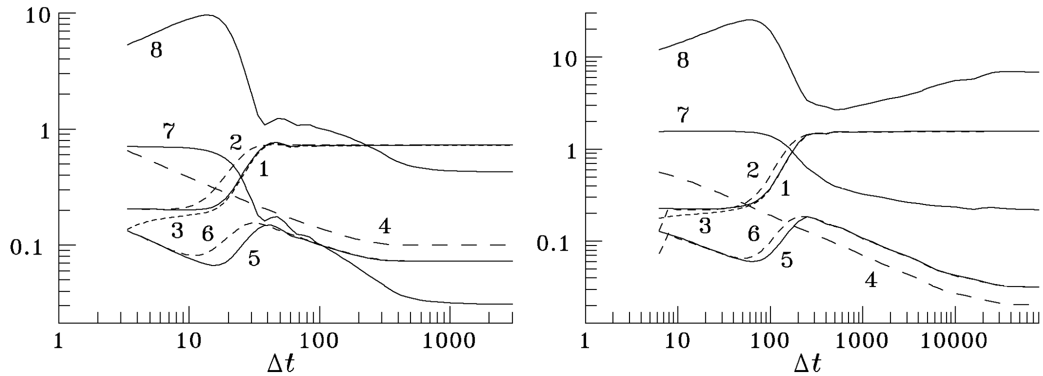

Fig. 6. Dependence of the mean characteristics of the "wp" fit on Δt for model harmonic wave with noise (left, $n = 300$, $P = 30$) and visual observations of RT Cyg (right, $n = 7154$, $P = 190^{\rm d}$): 1–σ_1; 2–σ_2; 3–σ_3; 4–R; 5–$R\sigma_1$; 6–$R\sigma_2$; 7–σ_c; 8–$S/N = \sigma_c/(Rs_1)$

We propose to use the value of Δt which corresponds to the maximum of the ratio "signal/noise". This procedure may be illustrated by Fig. 6. For numerical study we have used two data sets. The first one is an artificial one defined at times $t_k = 1, 2, \cdot, 300$ with signal values being a superposition of pure sine of unit amplitude and period $P = 30$ with normally distributed noise with rms deviation 0.2. The second set contained $n = 7154$ visual observations of the Mira–type star RT Cyg obtained by the members of AFOEV and photographic data from the Odessa plate collection (Marsakova et al. 1997). Both sets were reduced by using the same program.

With increasing Δt, the values of σ_1 and σ_2 remain nearly the same until some value when systematic differences of the fit from the true shape become significant. One may note that σ_2 becomes significantly larger than σ_1. This may be interpreted by the fact that one uses the sum $\Phi_t + \Phi_d$ instead of Φ_d to estimate the mean value of σ_2, whereas the deviation of the central point of the local fit $\varphi(t_0, t_0, \Delta t)$ from the true shape is smaller than of the whole fit. For larger Δt these both estimates coincide at the higher level as the fit does not response to periodic variations. The parameter σ_3 is smaller than σ_1 because it does not take into account the expression in brackets in the right side of Eq. (22) and thus is biased. This difference is significant for small Δt, when the number of the data inside the subinterval is small and decreases with increasing Δt. The parameter $\sum_{\alpha\beta=0}^{m} R_{\alpha\beta} f_\alpha(0) f_\beta(0)$ (Eq. 21) is equal to R_{00} for the "wp" fit. Its mean (over all data) value R^2 is shown by line "4" in Fig. 6. The parameter R decreases with Δt nearly proportionally to $\Delta t^{-1/2}$ because the number of the data in the subinterval increases proportionally to Δt. For large Δt, all data are involved in the local fit, thus R is not dependent on Δt and only may see a standstill. Accuracy estimates of the fit $R\sigma_1$ and $R\sigma_2$ behave in a more complex way. At first they decrease with Δt, as σ_1 remain constant and R decreases. Then their increase becomes more significant than decrease of R and the product $R\sigma_1$ increases, reaches its maximum and continues to decrease because of the next standstill of σ_1 and decrease of R. The standstill of $R\sigma_1$ occurs when R has its standstill. Thus one may conclude that the minimum value $R\sigma_1$ corresponds to $\Delta t \to \infty$, i.e. the error estimate is the best if we use the global fit instead of local and approximate the signal by polynomial of order m. This trivial situation needs no local fits for different Δt at all. However, if we are interested in the cyclic variations, we may choose Δt corresponding to local minimum of $R\sigma_1$.

Similarly behaves $R\sigma_2$, but this value is overestimated in the interval of Δt we are interested in and thus has no practical meaning.

As the characteristic of the amplitude of the fit one may choose the rms deviation σ_c of the smoothed values from the mean. Its dependence on Δt is shown by line "7" in Fig. 6. For small Δt, when systematic differences are small, it has a standstill followed by an abrupt decrease. From σ_c and $R\sigma_1$ we may combine a parameter $\sigma_c/(R\sigma_1)$ which we call "signal/noise" (S/N) ratio. The position of its maximum may be used for determination of the optimal value of Δt. It is slightly smaller than that (Δt_0) obtained from the minimum of $R\sigma_1$ because of decrease of R, but practically this difference does not exceed 10 per cent and may be used for control of the value Δt_0.

The position of the maximum of S/N for model harmonic signal is in good agreement with that obtained for continuous approximation (Eq. (84) and the following paragraph). For RT Cyg the value Δt_0 is smaller than the expected one $0.5450P$ because the shape of the light curve is not sine–like and may be described by 3-harmonic fit (Marsakova et al. 1997).

Determination of the optimal value of Δt needs more computational time than for the fit with fixed Δt. For each data set one will obtain different values. However, one should recommend to use the same Δt for all runs not to change spectral properties of the fit (e.g. Tremko et al. 1996). For this purpose one may extend the summation from one run to all runs or to use some value close to the mean for different runs.

9. Conclusions

There are two obvious differences from the classical running mean. At first, one approximates data within the interval $[t_0 - \Delta t, t_0 + \Delta t]$ by a parabola instead of a constant (mean), thus parabolic and cubic variations would be fit exactly for observations equidistantly distributed in time. This method is efficient also for data with gaps, but in this case only a parabolic signal will be fit exactly.

At second, by using the coefficients p_i, one would obtain a smoothing function which is continuous at all t as well as its first derivative. Second and higher-order derivatives are discontinuous at $2n$ points $t = t_i \pm \Delta t$. At these points a running mean function is discontinuous itself, as well as *all* derivatives. Thus one may say that a running mean fit is a spline of order 1 and defect 1, a running parabola fit is a spline of order 6 and defect 5.

For evenly spaced data one often uses a local approximation at times of observations (cf. Whittaker & Robinson 1928) and neglecting intermediate arguments. In this case, discontinuity of the smoothing function is not important and thus unweighted parabolae may be preferred to make smaller accuracy estimates. However, discontinuous smoothing curve may not allow determination of a true extremum, being highly affected by statistical errors of the signal. Moreover, the line interpolating of the smoothed values may not coincide with the smoothing function at the intermediate arguments.

For fixed filter half-width Δt, the accuracy estimate is the best for "um". However, this fit has the worst spectral characteristics and the largest systematic differences from smooth curves. Thus one has to determine a value of Δt optimizing the balance between the systematic and statistical errors of the fits.

If the signal values are evenly distributed in time and their number per period is large enough, one may use the "continuous limit" discussed here in detail. For arbitrary distribution of observations in time one may use the derived precise analytic expressions and to determine parameter Δt optimizing the preferred papameter(s) of the fit.

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